

Quiver \mathcal{D} -Modules and the Riemann-Hilbert Correspondence

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Abstract

In this paper, we show that every regular singular \mathcal{D} -module in \mathbb{C}^n whose singular locus is a normal crossing is isomorphic to a quiver \mathcal{D} -module – a \mathcal{D} -module whose definition is based on certain representations of the hypercube quiver. To be more precise we give an equivalence of the respective categories. Our definition of quiver \mathcal{D} -modules is based on the one of Khoroshkin and Varchenko. To prove the equivalence, we use an equivalence by Galligo, Granger and Maisonobe for regular singular \mathcal{D} -modules whose singular locus is a normal crossing which involves the classical Riemann-Hilbert correspondence.

The classical version of the Riemann-Hilbert correspondence, as it was proven independently by Masaki Kashiwara [Kas84] and Zoghman Mebkhout [Meb84] in 1980, yields an equivalence $\text{Mod}_{\text{rh}}(\mathcal{D}_X) \xrightarrow{\cong} \text{Perv}(X)$ between regular holonomic \mathcal{D}_X -modules and perverse sheaves on X . In dimension one locally at 0 other equivalences are known which one might obtain from the classical version: The equivalence of the category $\text{Mod}_{\text{rh}}(\mathcal{D})$ of regular holonomic \mathcal{D} -modules with the category \mathcal{C}_1 and the category $\text{Qui}_1^{\Sigma_1}$, the categories of finite quiver representations $E \xleftarrow[u]{v} F$ over \mathbb{C} fulfilling that $u \circ v + \text{Id}$ is invertible and $\text{Spec}(u \circ v) \subset \Sigma_1$, respectively (see [Mal91], [Bjö93] or [Dim04]). In higher dimension André Galligo, Michel Granger and Philippe Maisonobe proved that in the case of a normal crossing divisor in dimension n , the category of perverse sheaves with respect to the induced normal crossing stratification is equivalent to the category \mathcal{C}_n (the generalization of \mathcal{C}_1). This means, using the Riemann-Hilbert correspondence, that the category of regular holonomic \mathcal{D} -modules whose singular locus is a normal crossing is equivalent to \mathcal{C}_n (see [GGM85a] and [GGM85b]). However, it is not that easy to assign a \mathcal{D} -module to a given quiver representation with respect to this equivalence concretely.

A contribution to the question of how to assign to a quiver representation a \mathcal{D} -module comes from Sergei Khoroshkin and Alexander Varchenko [KV06]. To a given hyperplane arrangement in \mathbb{C}^n , they associate a quiver. And to each finite representation over \mathbb{C} of such a quiver, they associate a \mathcal{D} -module in a rather intuitive way. This yields a functor E from the category of representations over these quivers into the category of holonomic \mathcal{D} -modules. Using this definition in dimension one, one sees that this gives a functor from $\text{Qui}_1^{\Sigma_1}$ to $\text{Mod}_{\text{rh}}(\mathcal{D})$ one can use as quasi-inverse for the equivalence above. In particular one sees that every regular holonomic \mathcal{D} -module in dimension one locally at 0 is isomorphic to a quiver \mathcal{D} -module. This makes their construction very promising for higher dimensions.

The main idea of our work is to use this construction of quiver \mathcal{D} -modules by Khoroshkin and Varchenko in the case of a normal crossing hyperplane arrangement and to combine it with the

theorem of Galligo, Granger and Maisonobe. In Section 1 we give some general statements on representations of the hypercube quiver. In Section 2 we define the category of quiver \mathcal{D} -modules and give their main properties. In our Main Theorem 2.7 we give the link between this category and the category $Qui_n^{\Sigma_1}$ using the theorem of Galligo, Granger and Maisonobe.

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1 Finite representations of the hypercube quiver

In the following let us consider finite representations over \mathbb{C} of the following quiver:

Let $n \in \mathbb{N}^+$ and let $\mathcal{P}(\{1, \dots, n\})$ denote the power set of $\{1, \dots, n\}$. The quiver consists of 2^n vertices which we denote by $I \in \mathcal{P}(\{1, \dots, n\})$, and $n2^n$ oriented edges

$$I \longleftrightarrow I \cup \{i\}$$

for $i \in \{1, \dots, n\} \setminus I$. This gives us a kind of hypercube quiver by imaging that the vertices of the quiver lie on the vertices of a n -dimensional hypercube, and we have two edges exactly for every edge of the hypercube.

1.1 Definitions and basic properties

In the following we are going to define three standard categories of hypercube quiver representations, denoted Qui_n , \mathcal{C}_n and $Qui_n^{\Sigma_1}$. Let us start with the definition of Qui_n . This is basically just the category of finite representations over \mathbb{C} of the above hypercube quiver.

Definition 1.1. *The category Qui_n for $n \in \mathbb{N}^+$ is defined as follows:*

- *The objects consist of 2^n finitely generated \mathbb{C} -vector spaces denoted V_I where $I \in \mathcal{P}(\{1, \dots, n\})$, equipped with $n2^n$ linear maps $u_{I,i}$ and $y_{I,i}$ for $i \in \{1, \dots, n\} \setminus I$,*

$$V_I \xleftarrow{u_{I,i}} \xrightarrow{y_{I,i}} V_{I \cup \{i\}}$$

and they satisfy the following commutativity conditions for $i, j \in \{1, \dots, n\} \setminus I$:

$$\begin{aligned} u_{I \cup \{i\}, j} \circ u_{I, i} &= u_{I \cup \{j\}, i} \circ u_{I, j} & y_{I, i} \circ y_{I \cup \{i\}, j} &= y_{I, j} \circ y_{I \cup \{j\}, i} \\ y_{I \cup \{i\}, j} \circ u_{I \cup \{j\}, i} &= u_{I, i} \circ y_{I, j} \end{aligned}$$

- *A morphism between two objects $(V_I, u_{I,i}, y_{I,i})$ and $(V'_I, u'_{I,i}, y'_{I,i})$ is given by 2^n linear maps $h_I: V_I \rightarrow V'_I$ such that $u'_{I,i} \circ h_I = h_{I \cup \{i\}} \circ u_{I,i}$ and $h_I \circ y_{I,i} = y'_{I,i} \circ h_{I \cup \{i\}}$ for $i \in \{1, \dots, n\} \setminus I$.*

Now, let us define the categories \mathcal{C}_n and $\text{Qui}_n^{\Sigma_1}$. They are full subcategories of Qui_n fulfilling an additional constraint on their objects.

Definition 1.2. *The category \mathcal{C}_n is the full subcategory of Qui_n such that every object $(V_I, u_{I,i}, y_{I,i})$ additionally fulfils that $u_{I,i} \circ y_{I,i} + \text{Id}_{V_{I \cup \{i\}}}$ and $y_{I,i} \circ u_{I,i} + \text{Id}_{V_I}$ are invertible.*

Definition 1.3. *The category $\text{Qui}_n^{\Sigma_1}$ is the full subcategory of Qui_n such that every object $(V_I, u_{I,i}, y_{I,i})$ additionally fulfils that $\text{Spec}(u_{I,i} \circ y_{I,i}), \text{Spec}(y_{I,i} \circ u_{I,i}) \subset \Sigma_1 := \Sigma + 1$ where*

$$\Sigma := \left\{ \alpha \in \mathbb{C} \mid \begin{array}{ll} \geq 0 & \text{if } \text{Re}(\alpha) = -1, \\ < 0 & \text{if } \text{Re}(\alpha) = 0, \\ \text{arbitrary} & \text{otherwise.} \end{array} \right\}$$

We note that $u_{I,i} \circ y_{I,i} + \text{Id}_{V_{I \cup \{i\}}}$ is invertible iff $y_{I,i} \circ u_{I,i} + \text{Id}_{V_I}$ is invertible, and $\text{Spec}(u_{I,i} \circ y_{I,i}) \subset \Sigma_1$ iff $\text{Spec}(y_{I,i} \circ u_{I,i}) \subset \Sigma_1$.

The last topic in this subsection is a dualizing functor acting on our quiver categories.

Definition/Proposition 1.4. *The contravariant functor $D: \text{Qui}_n \rightarrow \text{Qui}_n$ is defined on objects $(V_I, u_{I,i}, y_{I,i})$ of Qui_n by*

$$D \left(V_I \begin{array}{c} \xleftarrow{u_{I,i}} \\ \xrightarrow{y_{I,i}} \end{array} V_{I \cup \{i\}} \right) := V_I^* \begin{array}{c} \xleftarrow{y_{I,i}^*} \\ \xrightarrow{u_{I,i}^*} \end{array} V_{I \cup \{i\}}^*$$

where V_I^* is the dual vector space of V_I and $u_{I,i}^*, y_{I,i}^*$ are the dual/transpose maps of $u_{I,i}, y_{I,i}$. Let (h_I) denote a morphism in Qui_n . Then we set

$$D((h_I)) := (h_I^*)$$

where h_I^* is the dual map of h_I . This yields an equivalence of categories where D is its own quasi-inverse, and it also establishes an equivalence from $\text{Qui}_n^{\Sigma_1}$ to $\text{Qui}_n^{\Sigma_1}$, and from \mathcal{C}_n to \mathcal{C}_n as well.

1.2 An equivalence of categories

The goal of this subsection is to prove in Theorem 1.6 an equivalence (or rather an isomorphism) of the categories \mathcal{C}_n and $\text{Qui}_n^{\Sigma_1}$. This is a principal component of the present work. We use the following pair of functors:

Definition 1.5. *The covariant functors $Q: \text{Qui}_n^{\Sigma_1} \rightarrow \mathcal{C}_n$ and $\mathcal{G}: \mathcal{C}_n \rightarrow \text{Qui}_n^{\Sigma_1}$ are defined by: Let $(V_I, u_{I,i}, c_{I,i})$ denote an object in $\text{Qui}_n^{\Sigma_1}$ and let (h_I) denote a morphism in $\text{Qui}_n^{\Sigma_1}$. We set*

$$Q((V_I, u_{I,i}, c_{I,i})) := (V_I, u_{I,i}, y_{I,i}) \quad \text{and} \quad Q((h_I)) := (h_I)$$

where $y_{I,i} := \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} (c_{I,i} \circ u_{I,i})^{k-1} \circ c_{I,i} = c_{I,i} \circ \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} (u_{I,i} \circ c_{I,i})^{k-1}$.

Let $(V_I, u_{I,i}, w_{I,i})$ denote an object in \mathcal{C}_n and let (h_I) denote a morphism in \mathcal{C}_n . Then, set

$$\mathcal{G}((V_I, u_{I,i}, w_{I,i})) := (V_I, u_{I,i}, x_{I,i}) \quad \text{and} \quad \mathcal{G}((h_I)) := (h_I).$$

The map $x_{I,i}$ is given as follows: Let $s_{I,i}: V_I \rightarrow V_I$ denote the unique linear map with eigenvalues in Σ_1 such that

$$e^{2\pi i s_{I,i}} = w_{I,i} \circ u_{I,i} + \text{Id}_{V_I} \quad \text{and set} \quad x_{I,i} := \left(\sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} s_{I,i}^{k-1} \right)^{-1} \circ w_{I,i}.$$

Theorem 1.6. *The category \mathcal{C}_n is isomorphic to the category $\text{Qui}_n^{\Sigma_1}$ using the pair of covariant functors $Q: \text{Qui}_n^{\Sigma_1} \rightarrow \mathcal{C}_n$ and $\mathcal{G}: \mathcal{C}_n \rightarrow \text{Qui}_n^{\Sigma_1}$.*

Note that i in $2\pi i$ is the imaginary unit. Before proving the theorem, we verify two helpful statements from matrix analysis.

Proposition 1.7. *Let E denote a finite dimensional \mathbb{C} -vector space and let $f: E \rightarrow E$ denote an invertible linear map. Then there exists a unique linear map $g: E \rightarrow E$ with $\text{Spec}(g) \subset \Sigma_1$ such that*

$$f = e^{2\pi i g}.$$

Proof. We choose the branch of the logarithm defined on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ with image contained in $\{\alpha \in \mathbb{C} \mid 0 < \text{Im}(\alpha) < 2\pi\}$, and use a unique extension to $\mathbb{C} \setminus \{0\}$ with image in $2\pi i \Sigma_1$. Note that every complex number has a unique representative in this strip up to $\pm 2\pi i \mathbb{N}$. The existence and uniqueness of $2\pi i g$ follows now with the aid of [HJ91, Corollary 6.2.12] which deals with finding a matrix A for a given invertible matrix B such that $e^A = B$. \square

The next corollary will be auxiliary to prove commutativity conditions later.

Corollary 1.8. *Let E, F denote two finite dimensional \mathbb{C} -vector spaces and let $\gamma: E \rightarrow F$ denote a linear map. Furthermore, let $\alpha: E \rightarrow E$ and $\beta: F \rightarrow F$ denote two linear maps with $\text{Spec}(\alpha), \text{Spec}(\beta) \subset \Sigma_1$. Then:*

$$\gamma \circ e^{2\pi i \alpha} = e^{2\pi i \beta} \circ \gamma \iff \gamma \circ \alpha = \beta \circ \gamma$$

Proof. We only need to prove the direction “ \Rightarrow ”. This proof is divided into three parts:

(i) Assume that $\gamma: E \rightarrow F$ is invertible. We receive

$$\gamma \circ e^{2\pi i \alpha} = e^{2\pi i \beta} \circ \gamma \iff e^{2\pi i \beta} = e^{2\pi i (\gamma \circ \alpha \circ \gamma^{-1})}.$$

The eigenvalues of β and $\gamma \circ \alpha \circ \gamma^{-1}$ are both contained in Σ_1 . Using the uniqueness given in Proposition 1.7, we receive the claimed equality.

(ii) To prove the general case, we need the following small statement:

$$\alpha \text{ preserves a linear subspace } \tilde{E} \text{ of } E, \text{ i. e. } \alpha(\tilde{E}) \subset \tilde{E}. \iff e^{2\pi i \alpha} \text{ preserves } \tilde{E}.$$

The direction “ \Rightarrow ” is clear. To prove “ \Leftarrow ”, use [HJ91, Corollary 6.2.12] to receive that α is a polynomial in $e^{2\pi i\alpha}$ (the concrete form of the polynomial depends on the map α).

- (iii) Now, let us prove the general case. Examining $\gamma \circ e^{2\pi i\alpha} = e^{2\pi i\beta} \circ \gamma$, we see that $e^{2\pi i\beta}$ preserves $\text{im}(\gamma)$ and that $e^{2\pi i\alpha}$ preserves $\ker(\gamma)$. By part (ii), $\text{im}(\gamma)$ and $\ker(\gamma)$ are preserved by β and α , respectively. This gives us well-defined maps

$$\overline{e^{2\pi i\alpha}}: E/\ker(\gamma) \rightarrow E/\ker(\gamma) \quad \text{and} \quad \overline{\alpha}: E/\ker(\gamma) \rightarrow E/\ker(\gamma)$$

where $\overline{e^{2\pi i\alpha}} = e^{2\pi i\overline{\alpha}}$. Consider the following diagram whose first and last square commute:

$$\begin{array}{ccccccc} E & \twoheadrightarrow & E/\ker(\gamma) & \xrightarrow{\overline{\gamma}} & \text{im}(\gamma) & \hookrightarrow & F \\ \alpha \downarrow & & \overline{\alpha} \downarrow & & \beta \downarrow & & \beta \downarrow \\ E & \twoheadrightarrow & E/\ker(\gamma) & \xrightarrow{\overline{\gamma}} & \text{im}(\gamma) & \hookrightarrow & F \end{array}$$

One has $\text{Spec}(\overline{\alpha}) \subset \text{Spec}(\alpha) \subset \Sigma_1$. By part (i) we receive now that the commutativity of

$$\begin{array}{ccc} E/\ker(\gamma) & \xrightarrow{\overline{\gamma}} & \text{im}(\gamma) \\ e^{2\pi i\overline{\alpha}} \downarrow & & e^{2\pi i\beta} \downarrow \\ E/\ker(\gamma) & \xrightarrow{\overline{\gamma}} & \text{im}(\gamma) \end{array} \quad \text{implies that} \quad \begin{array}{ccc} E/\ker(\gamma) & \xrightarrow{\overline{\gamma}} & \text{im}(\gamma) \\ \overline{\alpha} \downarrow & & \beta \downarrow \\ E/\ker(\gamma) & \xrightarrow{\overline{\gamma}} & \text{im}(\gamma) \end{array}$$

commutes. This yields the commutativity of the above diagram and proves the claim. \square

Proof of Theorem 1.6. For simplicity, we set for a linear map A

$$\psi(A) := \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} A^{k-1}.$$

We need to check several small statements to obtain the theorem:

- (i) Verify that $x_{I,i}$ is well-defined: The map $w_{I,i} \circ u_{I,i} + \text{Id}_{V_I}$ is invertible by assumption. The existence and uniqueness of $s_{I,i}$ follows from Proposition 1.7. As $\psi(0) = 2\pi i \neq 0$ and

$$\psi(\lambda) = \frac{e^{2\pi i\lambda} - 1}{\lambda} \neq 0$$

for $\lambda \in \Sigma_1 \setminus \{0\}$, we receive that the eigenvalues of $\psi(s_{I,i})$ are non-zero and therefore $x_{I,i} = \psi(s_{I,i})^{-1} \circ w_{I,i}$ is well-defined.

- (ii) We have to check that $Q: \mathcal{Q}ui_n^{\Sigma_1} \rightarrow \mathcal{C}_n$ is a well-defined functor:

Let $(V_I, u_{I,i}, c_{I,i})$ denote an object in $\mathcal{Q}ui_n^{\Sigma_1}$ and let $(V_I, u_{I,i}, y_{I,i})$ denote its image under Q . This is indeed an object in \mathcal{C}_n : The map

$$y_{I,i} \circ u_{I,i} + \text{Id}_{V_I} = \psi(c_{I,i} \circ u_{I,i}) \circ c_{I,i} \circ u_{I,i} + \text{Id}_{V_I} = e^{2\pi i(c_{I,i} \circ u_{I,i})}$$

is obviously invertible. The commutativity conditions in \mathcal{C}_n follow by direct computation from those in $\mathcal{Q}ui_n^{\Sigma_1}$. Now, let (h_I) denote a morphism from $(V_I, u_{I,i}, c_{I,i})$ to $(V'_I, u'_{I,i}, c'_{I,i})$ in $\mathcal{Q}ui_n^{\Sigma_1}$. To prove that $Q((h_I)) = (h_I)$ is a morphism in \mathcal{C}_n from $(V_I, u_{I,i}, y_{I,i})$ to $(V'_I, u'_{I,i}, y'_{I,i})$, we need to check the equations

$$u'_{I,i} \circ h_I = h_{I \cup \{i\}} \circ u_{I,i} \quad \text{and} \quad h_I \circ y_{I,i} = y'_{I,i} \circ h_{I \cup \{i\}}.$$

Both equations follow directly from the fact that (h_I) is a morphism in $\mathcal{Q}ui_n^{\Sigma_1}$.

(iii) We have to check that $\mathcal{G}: \mathcal{C}_n \rightarrow \mathcal{Q}ui_n^{\Sigma_1}$ is a well-defined functor:

Let $(V_I, u_{I,i}, w_{I,i})$ denote an object in \mathcal{C}_n and let $(V_I, u_{I,i}, x_{I,i})$ denote its image under \mathcal{G} . $(V_I, u_{I,i}, x_{I,i})$ is indeed an object in $\mathcal{Q}ui_n^{\Sigma_1}$:

- We have the equality

$$x_{I,i} \circ u_{I,i} = \psi(s_{I,i})^{-1} \circ w_{I,i} \circ u_{I,i} = \psi(s_{I,i})^{-1} \circ (e^{2\pi i s_{I,i}} - \text{Id}_{V_I}) = s_{I,i}$$

which gives us $\text{Spec}(x_{I,i} \circ u_{I,i}) \subset \Sigma_1$.

- To prove the commutativity conditions, we use the following identities in \mathcal{C}_n :

$$\begin{aligned} e^{2\pi i s_{I,j}} \circ w_{I,i} &= w_{I,i} \circ e^{2\pi i s_{I \cup \{i\},j}} \\ u_{I,i} \circ e^{2\pi i s_{I,j}} &= e^{2\pi i s_{I \cup \{i\},j}} \circ u_{I,i} \\ e^{2\pi i s_{I,i}} \circ e^{2\pi i s_{I,j}} &= e^{2\pi i s_{I,j}} \circ e^{2\pi i s_{I,i}} \end{aligned}$$

These yield with the aid of Corollary 1.8:

$$\begin{aligned} \star \quad s_{I,j} \circ w_{I,i} &= w_{I,i} \circ s_{I \cup \{i\},j} &\implies w_{I,i} \circ \psi(s_{I \cup \{i\},j})^{-1} &= \psi(s_{I,j})^{-1} \circ w_{I,i} \\ \star \quad s_{I,i} \circ s_{I,j} &= s_{I,j} \circ s_{I,i} &\implies \psi(s_{I,i})^{-1} \circ \psi(s_{I,j})^{-1} &= \psi(s_{I,j})^{-1} \circ \psi(s_{I,i})^{-1} \\ \star \quad u_{I,i} \circ s_{I,j} &= s_{I \cup \{i\},j} \circ u_{I,i} &\implies \psi(s_{I \cup \{i\},j})^{-1} \circ u_{I,i} &= u_{I,i} \circ \psi(s_{I,j})^{-1} \end{aligned}$$

The commutativity conditions follow now immediately.

Now, let (h_I) denote a morphism from $(V_I, u_{I,i}, w_{I,i})$ to $(V'_I, u'_{I,i}, w'_{I,i})$ in \mathcal{C}_n . To prove that $\mathcal{G}((h_I)) = (h_I)$ is a morphism from $(V_I, u_{I,i}, x_{I,i})$ to $(V'_I, u'_{I,i}, x'_{I,i})$ in $\mathcal{Q}ui_n^{\Sigma_1}$, we need to verify the identities

$$\begin{aligned} \star \quad u'_{I,i} \circ h_I &= h_{I \cup \{i\}} \circ u_{I,i} \\ \star \quad h_I \circ x_{I,i} &= x'_{I,i} \circ h_{I \cup \{i\}} \iff h_I \circ \psi(s_{I,i})^{-1} \circ w_{I,i} = \psi(s'_{I,i})^{-1} \circ h_I \circ w_{I,i}. \end{aligned}$$

The first one follows directly. To prove the second equation we use the equality

$$e^{2\pi i s'_{I,i}} \circ h_I = h_I \circ e^{2\pi i s_{I,i}}.$$

Now, Corollary 1.8 yields

$$s'_{I,i} \circ h_I = h_I \circ s_{I,i} \quad \text{and therefore} \quad h_I \circ \psi(s_{I,i})^{-1} = \psi(s'_{I,i})^{-1} \circ h_I.$$

- (iv) We show that $Q \circ \mathcal{G} = \text{Id}_{\mathcal{C}_n}$: For this we need to check for an object $(V_I, u_{I,i}, w_{I,i})$ in \mathcal{C}_n that $(Q \circ \mathcal{G})((V_I, u_{I,i}, w_{I,i})) = (V_I, u_{I,i}, w_{I,i})$. Let

$$\begin{aligned}\mathcal{G}((V_I, u_{I,i}, w_{I,i})) &=: (V_I, u_{I,i}, x_{I,i}) \quad \text{where } x_{I,i} = \psi(s_{I,i})^{-1} \circ w_{I,i} \quad \text{and} \\ Q((V_I, u_{I,i}, x_{I,i})) &=: (V_I, u_{I,i}, y_{I,i}) \quad \text{where } y_{I,i} = \psi(x_{I,i} \circ u_{I,i}) \circ x_{I,i}.\end{aligned}$$

By part (iii) of the proof, $x_{I,i}$, $u_{I,i}$ and $s_{I,i}$ fulfil the equality $x_{I,i} \circ u_{I,i} = s_{I,i}$. Using the definition of $x_{I,i}$, this yields

$$y_{I,i} = \psi(x_{I,i} \circ u_{I,i}) \circ x_{I,i} = \psi(s_{I,i}) \circ \psi(s_{I,i})^{-1} \circ w_{I,i} = w_{I,i}.$$

- (iv) We show that $\mathcal{G} \circ Q = \text{Id}_{\mathcal{Q}ui_n^{\Sigma_1}}$: We need to verify for an object $(V_I, u_{I,i}, c_{I,i})$ in $\mathcal{Q}ui_n^{\Sigma_1}$ that $(\mathcal{G} \circ Q)((V_I, u_{I,i}, c_{I,i})) = (V_I, u_{I,i}, c_{I,i})$. We set

$$\begin{aligned}Q((V_I, u_{I,i}, c_{I,i})) &=: (V_I, u_{I,i}, y_{I,i}) \quad \text{where } y_{I,i} = \psi(c_{I,i} \circ u_{I,i}) \circ c_{I,i} \quad \text{and} \\ \mathcal{G}((V_I, u_{I,i}, y_{I,i})) &=: (V_I, u_{I,i}, x_{I,i}) \quad \text{where } x_{I,i} = \psi(s_{I,i})^{-1} \circ y_{I,i}.\end{aligned}$$

We have the equality

$$e^{2\pi i(c_{I,i} \circ u_{I,i})} = y_{I,i} \circ u_{I,i} + \text{Id}_{V_I} = e^{2\pi i s_{I,i}}.$$

The eigenvalues of $c_{I,i} \circ u_{I,i}$ and $s_{I,i}$ are contained in Σ_1 . Thus, the uniqueness of $s_{I,i}$ (cf. Proposition 1.7) yields $c_{I,i} \circ u_{I,i} = s_{I,i}$. Hence,

$$x_{I,i} = \psi(s_{I,i})^{-1} \circ y_{I,i} = \psi(c_{I,i} \circ u_{I,i})^{-1} \circ \psi(c_{I,i} \circ u_{I,i}) \circ c_{I,i} = c_{I,i}.$$

All in all, this shows that $Q: \mathcal{Q}ui_n^{\Sigma_1} \rightarrow \mathcal{C}_n$ and $\mathcal{G}: \mathcal{C}_n \rightarrow \mathcal{Q}ui_n^{\Sigma_1}$ are inverse functors to each other and therefore they define an isomorphism between the categories $\mathcal{Q}ui_n^{\Sigma_1}$ and \mathcal{C}_n . \square

2 Quiver \mathcal{D} -modules in \mathbb{C}^n whose singular locus is a normal crossing

From now on $\mathcal{O} = \mathcal{O}_X$ will always denote the sheaf of analytic functions on $X = \mathbb{C}^n$ for a fixed integer $n \in \mathbb{N}^+$, and $\mathcal{D} = \mathcal{D}_X$ denotes the sheaf of rings of linear partial differential operators with analytic coefficients. Furthermore, we denote by z_1, \dots, z_n the coordinates of \mathbb{C}^n and by $\partial_i = \frac{\partial}{\partial z_i}$ the i -th partial derivation operator for $i = 1, \dots, n$.

2.1 Definitions and basic properties

Let us first define objects of quiver \mathcal{D} -modules. These are \mathcal{D} -modules defined in a very natural manner on the basis of certain quiver representations where we use the category $\mathcal{Q}ui_n$ as starting point. Our definition is based on the one in [KV06, Subsection 4.2] in the case of a normal crossing hyperplane arrangement whereas we use analytic \mathcal{D} -modules instead of algebraic ones.

Definition 2.1 (Variant of [KV06]). Let $\mathcal{V}_n = (V_I, B_{I \cup \{i\}, I}, B_{I, I \cup \{i\}})$ denote an object in the category $\mathcal{Q}ui_n$. We define the associated quiver \mathcal{D} -module $E\mathcal{V}_n$ as the quotient of

$$\bigoplus_{I \in \mathcal{P}(\{1, \dots, n\})} \left(\mathcal{D} \otimes_{\mathbb{C}} \overline{\Omega}_I \otimes_{\mathbb{C}} V_I \right)$$

by the subsheaf \mathcal{J} . The sections of \mathcal{J} over $U \subset \mathbb{C}^n$, open, are given by \mathbb{C} -linear combinations of the following elements

$$\begin{aligned} & a\partial_i \otimes_{\mathbb{C}} \omega_I \otimes_{\mathbb{C}} v_I - a \otimes_{\mathbb{C}} \omega_{I \cup \{i\}} \otimes_{\mathbb{C}} B_{I \cup \{i\}, I}(v_I) \quad \text{and} \\ & az_i \otimes_{\mathbb{C}} \omega_{I \cup \{i\}} \otimes_{\mathbb{C}} v_{I \cup \{i\}} - a \otimes_{\mathbb{C}} \omega_I \otimes_{\mathbb{C}} B_{I, I \cup \{i\}}(v_{I \cup \{i\}}) \end{aligned}$$

where $I \neq \{1, \dots, n\}$, $i \notin I$, $a \in \mathcal{D}(U)$, $v_J \in V_J$ for $J \in \mathcal{P}(\{1, \dots, n\})$ and $\overline{\Omega}_J := \{c\omega_J \mid c \in \mathbb{C}\}$ is a 1-dimensional \mathbb{C} -vector space generated by the element ω_J . The left \mathcal{D} -module structure on $E\mathcal{V}_n$ is given by left multiplication.

$\overline{\Omega}_J$ is used here to clarify to which summand of $\bigoplus_I \left(\mathcal{D} \otimes \overline{\Omega}_I \otimes V_I \right)$ an element belongs to. Our next aim is to receive a functor from $\mathcal{Q}ui_n$ into the category of \mathcal{D} -modules on \mathbb{C}^n .

Corollary 2.2. Let $\mathcal{V}_n = (V_I, B_{I \cup \{i\}, I}, B_{I, I \cup \{i\}})$ and $\mathcal{V}'_n = (V'_I, B'_{I \cup \{i\}, I}, B'_{I, I \cup \{i\}})$ denote two objects in $\mathcal{Q}ui_n$ and let

$$\phi = (h_I): \mathcal{V}_n \rightarrow \mathcal{V}'_n$$

denote a morphism from \mathcal{V}_n to \mathcal{V}'_n . Then ϕ induces naturally a morphism

$$E\phi: E\mathcal{V}_n \rightarrow E\mathcal{V}'_n.$$

Proof. Consider the following diagram whose rows are exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{J} & \hookrightarrow & \bigoplus_{I \in \mathcal{P}(\{1, \dots, n\})} \left(\mathcal{D} \otimes_{\mathbb{C}} \overline{\Omega}_I \otimes_{\mathbb{C}} V_I \right) & \twoheadrightarrow & E\mathcal{V}_n \longrightarrow 0 \\ & & \downarrow \tilde{h} & & \downarrow \bigoplus_I (\text{Id}_{\mathcal{D}} \otimes \text{Id}_{\overline{\Omega}_I} \otimes h_I) & & \downarrow \text{dashed} \\ 0 & \longrightarrow & \mathcal{J}' & \hookrightarrow & \bigoplus_{I \in \mathcal{P}(\{1, \dots, n\})} \left(\mathcal{D} \otimes_{\mathbb{C}} \overline{\Omega}_I \otimes_{\mathbb{C}} V'_I \right) & \twoheadrightarrow & E\mathcal{V}'_n \longrightarrow 0 \end{array}$$

(h_I) induces naturally a \mathcal{D} -linear map $\tilde{h}: \mathcal{J} \rightarrow \mathcal{J}'$ which fulfils on sections over $U \subset \mathbb{C}^n$, open,

$$\begin{aligned} & \tilde{h} \left(a\partial_i \otimes \omega_I \otimes v_I - a \otimes \omega_{I \cup \{i\}} \otimes B_{I \cup \{i\}, I}(v_I) \right) = \\ & = a\partial_i \otimes \omega_I \otimes h_I(v_I) - a \otimes \omega_{I \cup \{i\}} \otimes h_{I \cup \{i\}}(B_{I \cup \{i\}, I}(v_I)) = \\ & = a\partial_i \otimes \omega_I \otimes h_I(v_I) - a \otimes \omega_{I \cup \{i\}} \otimes B'_{I \cup \{i\}, I}(h_I(v_I)) \quad \text{and} \\ & \tilde{h} \left(az_i \otimes \omega_{I \cup \{i\}} \otimes v_{I \cup \{i\}} - a \otimes \omega_I \otimes B_{I, I \cup \{i\}}(v_{I \cup \{i\}}) \right) = \\ & = az_i \otimes \omega_{I \cup \{i\}} \otimes h_{I \cup \{i\}}(v_{I \cup \{i\}}) - a \otimes \omega_I \otimes h_I(B_{I, I \cup \{i\}}(v_{I \cup \{i\}})) = \\ & = az_i \otimes \omega_{I \cup \{i\}} \otimes h_{I \cup \{i\}}(v_{I \cup \{i\}}) - a \otimes \omega_I \otimes B'_{I, I \cup \{i\}}(h_{I \cup \{i\}}(v_{I \cup \{i\}})). \end{aligned}$$

This makes the square commute as indicated. In particular, it induces in a natural way a \mathcal{D} -linear morphism from $E\mathcal{V}_n$ to $E\mathcal{V}'_n$. \square

Proposition/Definition 2.3. *Let $\text{Mod}(\mathcal{D})$ denote the category of \mathcal{D} -modules on \mathbb{C}^n . Then we receive a covariant functor, denoted E , from the category Qui_n into the category of \mathcal{D} -modules*

$$E: \text{Qui}_n \rightarrow \text{Mod}(\mathcal{D}).$$

E associates to an object \mathcal{V}_n in Qui_n the object $E\mathcal{V}_n$ in $\text{Mod}(\mathcal{D})$ from Definition 2.1, and it associates to a morphism $\phi: \mathcal{V}_n \rightarrow \mathcal{V}'_n$ in Qui_n the morphism $E\phi: E\mathcal{V}_n \rightarrow E\mathcal{V}'_n$ in $\text{Mod}(\mathcal{D})$ from Corollary 2.2. The category of quiver \mathcal{D} -modules is the essential image of the functor E .

Proof. Let (h_I) and (g_I) denote two morphisms in Qui_n with compatible source and target, respectively. Then E preserves the composition of morphisms, using Corollary 2.2, as

$$(\text{Id}_{\mathcal{D}} \otimes \text{Id}_{\overline{\Omega}_I} \otimes h_I) \circ (\text{Id}_{\mathcal{D}} \otimes \text{Id}_{\overline{\Omega}_I} \otimes g_I) = (\text{Id}_{\mathcal{D}} \otimes \text{Id}_{\overline{\Omega}_I} \otimes (h_I \circ g_I)) \quad \text{and} \quad \widetilde{h \circ g} = \tilde{h} \circ \tilde{g}.$$

As E also preserves the identity morphism, E is indeed a functor from Qui_n to $\text{Mod}(\mathcal{D})$. \square

Now, we define the category $\text{Mod}_{\text{rh}}^S(\mathcal{D})$ of regular singular holonomic \mathcal{D} -modules whose singular locus is contained in the normal crossing. However, we have to mention that this denomination is a little bit sloppy as in fact the objects in $\text{Mod}_{\text{rh}}^S(\mathcal{D})$ need to fulfil a property on their characteristic variety from which the property on the singular locus follows.

Definition 2.4. *Let $S := \{z_1 \cdots z_n = 0\}$ denote the normal crossing in \mathbb{C}^n . S induces naturally a (Whitney) stratification of \mathbb{C}^n by 2^n disjoint strata $X_I \subset \mathbb{C}^n$ which are defined by*

$$\overline{X}_I := \{z_i = 0 \mid i \in I\}, \quad X_I := \overline{X}_I \setminus \left(\bigcup_{\substack{J \in \mathcal{P}(\{1, \dots, n\}) \\ \overline{X}_J \subsetneq \overline{X}_I}} \overline{X}_J \right)$$

for $I \in \mathcal{P}(\{1, \dots, n\})$. This fulfils $X_{\emptyset} = \mathbb{C}^n \setminus S$ and $S = \bigcup_{I, \dim X_I < n} X_I$.

The category $\text{Mod}_{\text{rh}}^S(\mathcal{D})$ is then defined to be the category of regular singular holonomic \mathcal{D} -modules whose characteristic variety is contained in

$$\bigcup_{I \in \mathcal{P}(\{1, \dots, n\})} T_{X_I}^* \mathbb{C}^n$$

where $T_{X_I}^ \mathbb{C}^n = \{(z, \xi) = (z_1, \dots, z_n, \xi_1, \dots, \xi_n) \in T^* \mathbb{C}^n \mid z \in X_I, \xi(v) = 0 \ \forall v \in TX_I\} =$*

$$= \{(z_1, \dots, z_n, \xi_1, \dots, \xi_n) \in T^* \mathbb{C}^n \mid z_i = 0 \Leftrightarrow i \in I, \xi_i = 0 \text{ for } i \notin I\}.$$

We note that

$$\bigcup_{I \in \mathcal{P}(\{1, \dots, n\})} T_{X_I}^* \mathbb{C}^n = \Delta_S$$

where $\Delta_S := \{(z_1, \dots, z_n, \xi_1, \dots, \xi_n) \in T^* \mathbb{C}^n \mid z_i \xi_i = 0 \text{ for all } i \in \{1, \dots, n\}\}$

which simplifies Definition 2.4. Thus, $\text{Mod}_{\text{rh}}^S(\mathcal{D})$ is the category of regular singular holonomic \mathcal{D} -modules whose characteristic variety is contained in Δ_S .

Let us explain how the property on the singular locus is implied by this: Let \mathcal{M} denote a holonomic \mathcal{D} -module and let $\text{Char}(\mathcal{M})$ denote its characteristic variety. The singular locus of \mathcal{M} is defined as the projection of $\text{Char}(\mathcal{M}) \setminus T_{\mathbb{C}^n}^* \mathbb{C}^n$ onto \mathbb{C}^n where $T_{\mathbb{C}^n}^* \mathbb{C}^n$ is the zero section of $T^* \mathbb{C}^n$. By direct computation one sees that the projection of $\Delta_S \setminus T_{\mathbb{C}^n}^* \mathbb{C}^n$ to \mathbb{C}^n is equal to $S = \bigcup_{I, \dim X_I < n} X_I$. Thus, the singular locus of \mathcal{M} is contained in S .

The stratification S of \mathbb{C}^n in fact also determines the characteristic variety of the quiver \mathcal{D} -modules as we will see now.

Theorem 2.5. *The functor E maps from the category Qui_n into the category $\text{Mod}_{\text{rh}}^S(\mathcal{D})$.*

Proof. Let $\mathcal{V}_n = (V_J, B_{J \cup \{j\}, J}, B_{J, J \cup \{j\}})$ denote an object in Qui_n . We define the good filtration $F_k E\mathcal{V}_n$ on $E\mathcal{V}_n$ as the filtration induced by the exact sequence

$$\mathcal{D} \otimes \left(\bigoplus_J \overline{\Omega}_J \otimes V_J \right) \twoheadrightarrow E\mathcal{V}_n \rightarrow 0$$

using the standard filtration $F_\bullet \mathcal{D}$ of \mathcal{D} . Recall that $F_k \mathcal{D}$ is the subsheaf of \mathcal{D} of differential operators of order at most $k \in \mathbb{Z}$. Set

$$\text{gr}_k^F E\mathcal{V}_n := F_k E\mathcal{V}_n / F_{k-1} E\mathcal{V}_n \quad \text{and} \quad \text{gr}^F \mathcal{D} := \bigoplus_k \text{gr}_k^F \mathcal{D} \cong \mathcal{O}_{\mathbb{C}^n}[\xi_1, \dots, \xi_n].$$

Let $P \in F_k \mathcal{D}(U)$ for $U \subset \mathbb{C}^n$, open, and $v_I \in V_I$ for $I \in \mathcal{P}(\{1, \dots, n\})$. We denote by $[P \otimes \omega_I \otimes v_I]$ the image of $P \otimes \omega_I \otimes v_I \in F_k \mathcal{D}(U) \otimes \overline{\Omega}_I \otimes V_I$ in $F_k E\mathcal{V}_n$. Furthermore, let $\sigma_k[P \otimes \omega_I \otimes v_I]$ be the image of $[P \otimes \omega_I \otimes v_I]$ in $\text{gr}_k^F E\mathcal{V}_n$. We will prove that $z_i \xi_i$ annihilates $\text{gr}_k^F E\mathcal{V}_n$ for every $k \in \mathbb{Z}$ and every $i \in \{1, \dots, n\}$. We need to distinguish two cases:

If $i \notin I$, then

$$z_i \xi_i \cdot \sigma_k[P \otimes \omega_I \otimes v_I] = \sigma_{k+1}[P z_i \partial_i \otimes \omega_I \otimes v_I] = \sigma_{k+1}[P \otimes \omega_I \otimes B_{I, I \cup \{i\}} B_{I \cup \{i\}, I}(v_I)] = 0.$$

If $i \in I$, then

$$z_i \xi_i \cdot \sigma_k[P \otimes \omega_I \otimes v_I] = \sigma_{k+1}[P \partial_i z_i \otimes \omega_I \otimes v_I] = \sigma_{k+1}[P \otimes \omega_I \otimes B_{I, I \setminus \{i\}} B_{I \setminus \{i\}, I}(v_I)] = 0.$$

In both cases $z_i \xi_i$ is an annihilator. Thus, the characteristic variety of $E\mathcal{V}_n$ is contained in Δ_S . This also shows us that the dimension of the characteristic variety of $E\mathcal{V}_n$ is at most $n = \dim_{\mathbb{C}} X$ and therefore $E\mathcal{V}_n$ is holonomic. As well, we see that $E\mathcal{V}_n$ is a regular holonomic \mathcal{D} -module using the fact that $(z_i \xi_i)^1$ is an annihilator of $\text{gr}^F E\mathcal{V}_n = \bigoplus_k \text{gr}_k^F E\mathcal{V}_n$ (cf. [Kas03, Definition 5.2]). \square

We note that in [KV06] a similar but slightly different proof of the holonomicity and the statement on the characteristic variety is given.

2.2 An equivalence with regular singular \mathcal{D} -modules in \mathbb{C}^n whose singular locus is a normal crossing

Let us clarify some notational facts: Let $\iota: U \hookrightarrow X$ denote the inclusion for an open subset U of \mathbb{C}^n . Then Γ_U is the functor which maps sheaves on \mathbb{C}^n to sheaves on \mathbb{C}^n defined by

$$\Gamma_U := \iota_* \iota^{-1}.$$

Moreover, let

$$\mathbb{C}^n = \prod_{i=1}^n \mathbb{C}_i \quad \text{and} \quad W_i := \mathbb{C}_i \setminus \mathbb{R}_0^+.$$

And for $I \in \mathcal{P}(\{1, \dots, n\})$ set

$$\Lambda_I := \sum_{k \in I} \Gamma_{\mathbb{C}_k \times \prod_{\substack{i=1 \\ i \neq k}}^n W_i} \mathcal{O} \quad \text{and} \quad \mathcal{O}_I := \frac{\Gamma_{\prod_{i=1}^n W_i} \mathcal{O}}{\Lambda_I}.$$

Note that $(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$ and $\Lambda_{I,0}$ are unitary, commutative rings w.r.t. addition and multiplication of functions. And $\mathcal{O}_{I,0}$ is an abelian additive group and a unitary left \mathcal{D}_0 -module. But $\mathcal{O}_{I,0}$ is not a ring, as in general the multiplication of functions is not well-defined.

The following theorem of A. Galligo, M. Granger and Ph. Maisonobe will be important for our computations:

Theorem 2.6 ([GGM85a], [GGM85b]). *The contravariant functor \mathcal{A} from $\text{Mod}_{rh}^S(\mathcal{D})$ to \mathcal{C}_n*

$$\mathcal{A}: \text{Mod}_{rh}^S(\mathcal{D}) \longrightarrow \mathcal{C}_n$$

$$\mathcal{M} \longmapsto \text{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{O}_{I,0}) \xrightleftharpoons[\text{var}_{I,i}]{\text{can}_{I,i}} \text{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{O}_{I,0})$$

establishes an equivalence of categories. $\text{can}_{I,i}$ is the canonical map or quotient map which sends solutions with values in $\mathcal{O}_{I,0}$ to solutions with values in $\mathcal{O}_{I \cup \{i\},0}$. $\text{var}_{I,i}$ is the variation around $z_i = 0$, i. e. we have

$$\text{var}_{I,i}(F) = M_i F - F \quad \text{for } F \in \text{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{O}_{I \cup \{i\},0})$$

where $M_i F$ is the class of a representative of F after analytic continuation around the axis $z_i = 0$. A \mathcal{D} -linear morphism $\phi: \mathcal{M} \rightarrow \mathcal{N}$ in $\text{Mod}_{rh}^S(\mathcal{D})$ is mapped under \mathcal{A} to the morphism

$$\left(\text{Hom}_{\mathcal{D}_{X,0}}(\phi_0, \mathcal{O}_{I,0}) \right) \text{ in } \mathcal{C}_n$$

where $\text{Hom}_{\mathcal{D}_{X,0}}(\phi_0, \mathcal{O}_{I,0}): \text{Hom}_{\mathcal{D}_{X,0}}(\mathcal{N}_0, \mathcal{O}_{I,0}) \rightarrow \text{Hom}_{\mathcal{D}_{X,0}}(\mathcal{M}_0, \mathcal{O}_{I,0})$ is given by $g \mapsto g \circ \phi_0$.

In their paper [GGM85a], Galligo, Granger and Maisonobe prove that the category of perverse sheaves in \mathbb{C}^n with respect to the normal crossing stratification Ξ is equivalent to the category of

quiver representations \mathcal{C}_n . They establish a functor

$$\alpha: \text{Perv}^\Xi(\mathbb{C}^n) \rightarrow \mathcal{C}_n.$$

Composing the functor α with the solution functor Sol one receives an equivalence of the categories $\text{Mod}_{\text{rh}}^S(\mathcal{D})$ and \mathcal{C}_n whereby the functor \mathcal{A} is naturally isomorphic to $\alpha \circ \text{Sol}$ [GGM85b].

Now, we are ready to state and prove the Main theorem:

Theorem 2.7. *The functors $\mathcal{A} \circ E$ and $Q \circ D$ are naturally isomorphic, i. e. the following diagram commutes up to a natural isomorphism*

$$\begin{array}{ccc} \text{Mod}_{\text{rh}}^S(\mathcal{D}) & \xrightarrow{\mathcal{A} \cong \alpha \circ \text{Sol}} & \mathcal{C}_n \\ E \uparrow & & \uparrow Q \\ \text{Qui}_n^{\Sigma_1} & \xrightarrow{D} & \text{Qui}_n^{\Sigma_1} \end{array}.$$

In particular, $E: \text{Qui}_n^{\Sigma_1} \rightarrow \text{Mod}_{\text{rh}}^S(\mathcal{D})$ is an equivalence of categories with quasi-inverse $D \circ \mathcal{G} \circ \mathcal{A}$, and $E \circ D \circ \mathcal{G}$ is a quasi-inverse of \mathcal{A} .

Furthermore, E is essentially surjective. This means the category of quiver \mathcal{D} -modules is exactly the category $\text{Mod}_{\text{rh}}^S(\mathcal{D})$, and every \mathcal{D} -module in $\text{Mod}_{\text{rh}}^S(\mathcal{D})$ is in fact isomorphic to a quiver \mathcal{D} -module as given in Definition 2.1.

In [KV06, Proposition 4.4] an equivalence of categories of quiver \mathcal{D} -modules in the case of a central arrangement of hyperplanes is also stated. But the essential image of the equivalence is not completely clarified. As domain they use a full subcategory of the category of representations over the quiver corresponding to the arrangement. This subcategory is defined by restricting the eigenvalues of several maps involved in the quiver representation. In the case of a normal crossing, this restriction is much more rigid than our restriction from Qui_n to $\text{Qui}_n^{\Sigma_1}$. This is a strong evidence that the essential image of their equivalence in our setting is not $\text{Mod}_{\text{h}}^S(\mathcal{D})$ or $\text{Mod}_{\text{rh}}^S(\mathcal{D})$.

The main parts of the proof of Theorem 2.7 are accomplished in Proposition 2.10 and Proposition 2.12. Before applying the functor \mathcal{A} to our quiver \mathcal{D} -modules we state some properties of $\mathcal{O}_{I,0}$ in Lemma 2.8 and Lemma 2.9 to simplify the arguments later.

Lemma 2.8. *For $I \in \mathcal{P}(\{1, \dots, n\})$ let \mathcal{O}_I as above. Then,*

- (i) z_j acts bijective on $\mathcal{O}_{I,0}$ if and only if $j \notin I$.
- (ii) ∂_j acts bijective on $\mathcal{O}_{I,0}$ if and only if $j \in I$.

Proof. Let $j \in \mathcal{P}(\{1, \dots, n\})$.

- (i) For $k = 1, \dots, n$ we use Z_k as dummy for \mathbb{C}_k or W_k . The inverse of z_j fulfils that $\frac{1}{z_j} \in (\Gamma_{\prod_{i=1}^n Z_i} \mathcal{O})_0$ if and only if $Z_j = W_j$. Thus z_j acts bijective on $(\Gamma_{\prod_{i=1}^n Z_i} \mathcal{O})_0$ if and only if $Z_j = W_j$. As $\Lambda_{I,0} = (\sum_{k \in I} \Gamma_{\mathbb{C}_k \times \prod_{i=1, i \neq k}^n W_i} \mathcal{O})_0$, we immediately see that z_j acts bijective on $\mathcal{O}_{I,0} \cong \frac{(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0}{\Lambda_{I,0}}$ if and only if $j \notin I$.

- (ii) Let $f(z_1, \dots, z_n) \in (\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$. As $\prod_{i=1}^n W_i$ is simply connected there exists a function $F(z_1, \dots, z_n) \in (\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$ such that $\partial_j F = f$. The other primitives of f w.r.t. ∂_j are given by $F(z_1, \dots, z_n) + C(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$, where $C \in (\Gamma_{\mathbb{C}_j \times \prod_{i=1, i \neq j}^n W_i} \mathcal{O})_0$ as C does not depend on z_j . Clearly, $j \in I$ if and only if for any such C it follows that $C \in \Lambda_{I,0}$. Now, we see that functions in $\mathcal{O}_{I,0}$ have a uniquely defined primitive w.r.t. ∂_j if and only if $j \in I$ (constants etc. move into the denominator of $\mathcal{O}_{I,0}$). \square

Lemma 2.9. *For every $r \in \{1, \dots, n\}$ fix a branch of the logarithm on $\mathbb{C}_r \setminus \mathbb{R}_{\geq 0}$. Let $M \in \mathbb{N}^+$ and let A denote a $M \times M$ -matrix with values in \mathbb{C} . We set*

$$z_r^A := \exp(A \cdot \ln(z_r)).$$

z_r^A is considered as a matrix with entries in $(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$ and all entries of z_r^A are invertible w.r.t. multiplication of functions in $(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$ and $(\Gamma_{\mathbb{C}_t \times \prod_{i=1, i \neq t}^n W_i} \mathcal{O})_0$ for $t \neq r$. Then:

- (i) The matrix z_r^A is invertible in $(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$ and $(\Gamma_{\mathbb{C}_t \times \prod_{i=1, i \neq t}^n W_i} \mathcal{O})_0$ for $t \neq r$.
- (ii) Let $I = \{m_1, \dots, m_{|I|}\} \in \mathcal{P}(\{1, \dots, n\})$ and $\{l_1, \dots, l_{n-|I|}\} = \{1, \dots, n\} \setminus I$. Assume we are given pairwise commuting $M \times M$ -matrices $A_{m_1}, \dots, A_{m_{|I|}}, A_{l_1}, \dots, A_{l_{n-|I|}}$ with values in \mathbb{C} , and the eigenvalues of $A_{m_1}, \dots, A_{m_{|I|}}$ lie in Σ . Let $\lambda = (\lambda_1, \dots, \lambda_M)^T \in \mathbb{C}^M$ and

$$\tilde{\mathcal{F}} := z_{l_1}^{A_{l_1}} \cdot \dots \cdot z_{l_{n-|I|}}^{A_{l_{n-|I|}}} \cdot z_{m_1}^{A_{m_1}} \cdot \dots \cdot z_{m_{|I|}}^{A_{m_{|I|}}}.$$

Then: $\partial_{m_1}^{-1} \dots \partial_{m_{|I|}}^{-1} \tilde{\mathcal{F}} \cdot \lambda \in (\Lambda_{I,0})^M \iff \lambda = (0, \dots, 0)^T$

Proof. (i) This becomes clear by passing to the Jordan normal form J of A . Let $\mu_1, \dots, \mu_q \in \mathbb{C}$ denote the eigenvalues of A . Then,

$$\det(\exp(A \cdot \ln(z_r))) = \det(\exp(J \cdot \ln(z_r))) = \prod_{i=1}^q (z_r^{\mu_i})^{p_i} \neq 0$$

where $p_1, \dots, p_q \in \mathbb{N}^+$. This yields the invertibility of $\exp(A \cdot \ln(z_r))$.

- (ii) We prove “ \Rightarrow ”. For simplicity let $I = \{1, \dots, |I|\}$. By part (i), the claim is equivalent to

$$\partial_1^{-1} z_1^{A_1} \cdot \dots \cdot \partial_{|I|}^{-1} z_{|I|}^{A_{|I|}} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix} \in (\Lambda_{I,0})^M.$$

This means, for $l = 1, \dots, |I|$, we find $f_l(z_1, \dots, z_n) \in \left((\Gamma_{\mathbb{C}_l \times \prod_{i=1, i \neq l}^n W_i} \mathcal{O})_0 \right)^M$ such that

$$\partial_1^{-1} z_1^{A_1} \cdot \dots \cdot \partial_{|I|}^{-1} z_{|I|}^{A_{|I|}} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix} = \sum_{l=1}^{|I|} f_l(z_1, \dots, z_n).$$

Let us apply $\text{var}_{|I|} \circ \dots \circ \text{var}_1$ to both sides of the equation where var_l was given by $M_l - \text{Id}$:

- Let us treat the (LHS): We receive

$$(\text{var}_{|I|} \circ \dots \circ \text{var}_1)(\text{LHS}) = \text{var}_1(\partial_1^{-1} z_1^{A_1}) \cdot \dots \cdot \text{var}_{|I|}(\partial_{|I|}^{-1} z_{|I|}^{A_{|I|}}) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix}.$$

Let us prove that $\text{var}_l(\partial_l^{-1} z_l^{A_l})$ is invertible in $(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$: We may pass to a single Jordan block J_a with eigenvalue $a \in \Sigma$ for our arguments. First, let $a \neq -1$. Then

$$\text{var}_l \left(\partial_l^{-1} \exp(J_a \ln(z_l)) \right) = (J_a + \text{Id})^{-1} \cdot \exp((J_a + \text{Id}) \ln(z_l)) \cdot (e^{2\pi i(J_a + \text{Id})} - \text{Id}).$$

Using part (i) and $\text{Spec}(J_a + \text{Id}) \subset \mathbb{C} \setminus \mathbb{Z}$, this is an invertible matrix in $(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$. Now, let $a = -1$. The matrix $\partial_l^{-1} \exp(J_{-1} \ln(z_l))$ is (up to a matrix which is independent of z_l) an upper-triangular matrix with $\ln(z_l)$ on the diagonal. Hence, the matrix $\text{var}_l(\partial_l^{-1} \exp(J_{-1} \ln(z_l)))$ is an upper-triangular matrix with $2\pi i$ as diagonal entry, and therefore it is invertible in $(\Gamma_{\prod_{i=1}^n W_i} \mathcal{O})_0$.

- Now, consider the (RHS): Using $M_1 f_1 = f_1$ and $M_1(\sum_{l=1}^{|I|} f_l) = M_1 f_1 + M_1(\sum_{l=2}^{|I|} f_l)$, we receive

$$\begin{aligned} (\text{var}_{|I|} \circ \dots \circ \text{var}_1)(\text{RHS}) &= (\text{var}_{|I|} \circ \dots \circ \text{var}_2) \left(M_1 \left(\sum_{l=1}^{|I|} f_l \right) - \sum_{l=1}^{|I|} f_l \right) = \\ &= (\text{var}_{|I|} \circ \dots \circ \text{var}_2) \left(M_1 \left(\sum_{l=2}^{|I|} f_l \right) - \sum_{l=2}^{|I|} f_l \right) = (\text{var}_{|I|} \circ \dots \circ \text{var}_1) \left(\sum_{l=2}^{|I|} f_l \right). \end{aligned}$$

As the variations commute on the left hand side (LHS), we receive furthermore

$$\begin{aligned} (\text{var}_{|I|} \circ \dots \circ \text{var}_1) \left(\sum_{l=2}^{|I|} f_l \right) &= (\text{var}_{|I|} \circ \dots \circ \text{var}_3 \circ \text{var}_1 \circ \text{var}_2) \left(\sum_{l=2}^{|I|} f_l \right) = \\ &= (\text{var}_{|I|} \circ \dots \circ \text{var}_3 \circ \text{var}_1 \circ \text{var}_2) \left(\sum_{l=3}^{|I|} f_l \right). \end{aligned}$$

Continuing this process, it yields $(\text{var}_{|I|} \circ \dots \circ \text{var}_1)(\text{RHS}) = (0, \dots, 0)^T$.

This leads to the equality

$$\text{var}_1(\partial_1^{-1} z_1^{A_1}) \cdot \dots \cdot \text{var}_{|I|}(\partial_{|I|}^{-1} z_{|I|}^{A_{|I|}}) \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The invertibility of all the matrices $\text{var}_l(\partial_l^{-1} z_l^{A_l})$ gives $\lambda_1 = \dots = \lambda_M = 0$ as claimed. \square

Now, let us consider how the quiver representation looks like after applying \mathcal{A} to a quiver \mathcal{D} -module.

Proposition 2.10. Let $\mathcal{V}_n = (V_I, B_{I \cup \{i\}, I}, B_{I, I \cup \{i\}})$ denote an object in $\mathcal{Q}ui_n^{\Sigma^1}$ with EV_n as corresponding quiver \mathcal{D} -module. Then, for every $I \in \mathcal{P}(\{1, \dots, n\})$, we are given a canonical isomorphism

$$\mathfrak{a}: V_I^* \xrightarrow{\cong} \text{Hom}_{\mathcal{D}_{X,0}}((EV_n)_0, \mathcal{O}_{I,0}).$$

Proof. We abbreviate $\mathcal{V} = \mathcal{V}_n$. The proof of this lemma will be carried out in several steps:

(i) We are given the following natural isomorphism:

$$\begin{aligned} \text{Hom}_{\mathcal{D}_0}((EV)_0, \mathcal{O}_{I,0}) &= \left\{ \phi \in \text{Hom}_{\mathcal{D}_0} \left(\bigoplus_J (\mathcal{D}_0 \otimes \overline{\Omega}_J \otimes V_J), \mathcal{O}_{I,0} \right) \mid \right. \\ &\quad \phi(\partial_j \otimes \omega_J \otimes v_J - 1 \otimes \omega_{J \cup \{j\}} \otimes B_{J \cup \{j\}, J}(v_J)) = 0, \\ &\quad \left. \phi(z_j \otimes \omega_{J \cup \{j\}} \otimes v_{J \cup \{j\}} - 1 \otimes \omega_J \otimes B_{J, J \cup \{j\}}(v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \dots, n\}, j \notin J \right\} \cong \\ &\cong \left\{ \bigoplus_J \phi^J \in \bigoplus_J \text{Hom}_{\mathbb{C}}(V_J, \mathcal{O}_{I,0}) \mid \partial_j \cdot \phi^J(v_J) - \phi^{J \cup \{j\}}(B_{J \cup \{j\}, J}(v_J)) = 0, \right. \\ &\quad \left. z_j \cdot \phi^{J \cup \{j\}}(v_{J \cup \{j\}}) - \phi^J(B_{J, J \cup \{j\}}(v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \dots, n\}, j \notin J \right\}. \end{aligned}$$

(ii) Consider the following system of equations from step (i)

$$\begin{aligned} \partial_j \cdot \phi^J - \phi^{J \cup \{j\}} \circ B_{J \cup \{j\}, J} &= 0 \\ z_j \cdot \phi^{J \cup \{j\}} - \phi^J \circ B_{J, J \cup \{j\}} &= 0 \end{aligned} \tag{*}$$

where $\phi^J \in \text{Hom}_{\mathbb{C}}(V_J, \mathcal{O}_{I,0})$ and $J \neq \{1, \dots, n\}$, $j \in \{1, \dots, n\} \setminus J$. We use the following algorithm (ALG) to express step by step every ϕ^K uniquely in terms of ϕ^I for $K \in \mathcal{P}(\{1, \dots, n\}) \setminus I$:

We can express $K \cup I$ as the union of the three disjoint sets $K_1 := K \cap I$, $K_2 := K \setminus K_1$, $K_3 := I \setminus K_1$. Then $K_2 \neq \emptyset$ or $K_3 \neq \emptyset$, as $K \neq I$. Lemma 2.8 yields that z_l, ∂_m act bijective on $\mathcal{O}_{I,0}$ for $l \in K_2$ and $m \in K_3$.

1. Step: If $K_2 = \emptyset$, skip this step. Otherwise we have for $l_1 \in K_2$:

$$z_{l_1} \cdot \phi^K - \phi^{K \setminus \{l_1\}} \circ B_{K \setminus \{l_1\}, K} = 0 \iff \phi^K = \frac{1}{z_{l_1}} \cdot \left(\phi^{K \setminus \{l_1\}} \circ B_{K \setminus \{l_1\}, K} \right)$$

For $l_2 \in K_2 \setminus \{l_1\}$ use the equation

$$z_{l_2} \cdot \phi^{K \setminus \{l_1\}} - \phi^{K \setminus \{l_1, l_2\}} \circ B_{K \setminus \{l_1, l_2\}, K \setminus \{l_1\}} = 0$$

to express ϕ^K in terms of $\phi^{K \setminus \{l_1, l_2\}}$. Continue until ϕ^K is expressed in terms of ϕ^{K_1} .

2. Step: If $K_3 = \emptyset$, we already expressed ϕ^K in terms of ϕ^I . Otherwise we have for $m_1 \in K_3$:

$$\partial_{m_1} \cdot \phi^{K_1} - \phi^{K_1 \cup \{m_1\}} \circ B_{K_1 \cup \{m_1\}, K_1} = 0 \iff \phi^{K_1} = \partial_{m_1}^{-1} \cdot \left(\phi^{K_1 \cup \{m_1\}} \circ B_{K_1 \cup \{m_1\}, K_1} \right)$$

For $m_2 \in K_3 \setminus \{m_1\}$ use the equation

$$\partial_{m_2} \cdot \phi^{K_1 \cup \{m_1\}} - \phi^{K_1 \cup \{m_1, m_2\}} \circ B_{K_1 \cup \{m_1, m_2\}, K_1 \cup \{m_1\}} = 0$$

to express ϕ^{K_1} in terms of $\phi^{K_1 \cup \{m_1, m_2\}}$. Continue until ϕ^{K_1} – and therefore ϕ^K – is expressed in terms of ϕ^I .

The order in which we solve for ϕ^I in (ALG) does not influence the result. This is ensured by the commutativity conditions on the maps $B_{\bullet, \bullet}$ and the fact that z_l, ∂_m commute for $l \notin I, m \in I$. Therefore every ϕ^K can be uniquely expressed in terms of ϕ^I .

So clearly $(\tilde{\star})$ implies that $\phi^I \in \text{Hom}_{\mathbb{C}}(V_I, \mathcal{O}_{I,0})$ fulfils the system

$$\begin{aligned} z_l \partial_l \cdot \phi^I - \phi^I \circ (B_{I, I \cup \{l\}} B_{I \cup \{l\}, I}) &= 0 \\ z_m \partial_m \cdot \phi^I - \phi^I \circ (B_{I, I \setminus \{m\}} B_{I \setminus \{m\}, I} - \text{Id}) &= 0 \end{aligned} \quad (\star)$$

of n equations where $l \notin I, m \in I$. On the other hand $(\tilde{\star})$ is likewise implied by (\star) using (ALG) as definition of ϕ^K for all $K \in \mathcal{P}(\{1, \dots, n\}) \setminus I$. This shows us that in fact

$$\begin{aligned} \text{Hom}_{\mathcal{D}_0}((E\mathcal{V})_0, \mathcal{O}_{I,0}) &\cong \left\{ \phi^I \in \text{Hom}_{\mathbb{C}}(V_I, \mathcal{O}_{I,0}) \mid z_l \partial_l \cdot \phi^I(v_I) - \phi^I(B_{I, I \cup \{l\}} B_{I \cup \{l\}, I}(v_I)) = 0, \right. \\ &\quad \left. z_m \partial_m \cdot \phi^I(v_I) - \phi^I((B_{I, I \setminus \{m\}} B_{I \setminus \{m\}, I} - \text{Id})(v_I)) = 0 \text{ for } l \in \{1, \dots, n\} \setminus I, m \in I \right\}. \quad (1) \end{aligned}$$

- (iii) The dimension of $\text{Hom}_{\mathcal{D}_0}((E\mathcal{V})_0, \mathcal{O}_{I,0})$ over \mathbb{C} is finite (see [GGM85b]). We use the following proposition of [GGM85b] to give an upper bound hereof:

Let $z_I^* = (z_1, \dots, z_n, \xi_1, \dots, \xi_n) \in T^*\mathbb{C}^n$ verifying $z_i \xi_i = 0$ for all i , and $z_i = 0 \Leftrightarrow i \in I$ and $\xi_i \neq 0 \Leftrightarrow i \in I$. Then $\dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}_0}((E\mathcal{V})_0, \mathcal{O}_{I,0}) = \text{mult}_{z_I^*} E\mathcal{V}$.

We use the definition of multiplicity as given in [GM93, Chapter V]. Supplementary, we use [Ser75, Subsection II.B)4]. As the definition becomes clear during the following computations, we do not repeat it here.

We use the good filtration on $E\mathcal{V}$ from the proof of Theorem 2.5. Its sections over $U \subset \mathbb{C}^n$, open, are given by

$$F_k E\mathcal{V}(U) = \frac{F_k \mathcal{D}(U) \otimes \left(\bigoplus_J \overline{\Omega}_J \otimes V_J \right)}{\left(F_k \mathcal{D}(U) \otimes \left(\bigoplus_J \overline{\Omega}_J \otimes V_J \right) \right) \cap \mathcal{J}(U)}$$

for $k \in \mathbb{N}_0$, and for $k \in \mathbb{Z} \setminus \mathbb{N}_0$ we have $F_k E\mathcal{V} = 0$. As before let

$$\text{gr}_k^F E\mathcal{V} = F_k E\mathcal{V} / F_{k-1} E\mathcal{V} \quad \text{and} \quad \text{gr}^F E\mathcal{V} = \bigoplus_{k \in \mathbb{N}_0} \text{gr}_k^F E\mathcal{V}.$$

Let $k \in \mathbb{N}_0$. Fix a point $\tilde{z}_I^* = (\tilde{z}_1, \dots, \tilde{z}_n, \tilde{\xi}_1, \dots, \tilde{\xi}_n) =: (\tilde{z}_I, \tilde{\xi}_I)$ where $\tilde{z}_i = 0 \Leftrightarrow i \in I$ and $\tilde{\xi}_i \neq 0 \Leftrightarrow i \in I$. Consider the stalk of $\text{gr}^F E\mathcal{V}$ at \tilde{z}_I . Set

$$M := (\text{gr}^F E\mathcal{V})_{\tilde{z}_I}.$$

z_i is an invertible element in $F_k \mathcal{D}_{\bar{z}_I}$ iff $i \notin I$. So let $i \notin I$ and $K \in \mathcal{P}(\{1, \dots, n\} \setminus \{i\})$. We denote by $[P \otimes \omega_{K \cup \{i\}} \otimes v_{K \cup \{i\}}]$ the image of $P \otimes \omega_{K \cup \{i\}} \otimes v_{K \cup \{i\}} \in F_k \mathcal{D}_{\bar{z}_I} \otimes \bar{\Omega}_{K \cup \{i\}} \otimes V_{K \cup \{i\}}$ in $(F_k EV)_{\bar{z}_I}$. We have the following identity in $(F_k EV)_{\bar{z}_I}$:

$$[P \otimes \omega_{K \cup \{i\}} \otimes v_{K \cup \{i\}}] = [z_i^{-1} P \otimes \omega_K \otimes B_{K, K \cup \{i\}}(v_{K \cup \{i\}})]$$

This allows us to “eliminate” all summands $[F_k \mathcal{D}_{\bar{z}_I} \otimes \bar{\Omega}_J \otimes V_J]$ in $(F_k EV)_{\bar{z}_I}$ with $J \setminus I \neq \emptyset$. Hence, we may assume that

$$(F_k EV)_{\bar{z}_I} = \frac{F_k \mathcal{D}_{\bar{z}_I} \otimes (\bigoplus_{J \subseteq I} \bar{\Omega}_J \otimes V_J)}{(F_k \mathcal{D}_{\bar{z}_I} \otimes (\bigoplus_{J \subseteq I} \bar{\Omega}_J \otimes V_J)) \cap \mathcal{J}_{\bar{z}_I}} \quad \text{or} \quad EV = \frac{\mathcal{D} \otimes (\bigoplus_{J \subseteq I} \bar{\Omega}_J \otimes V_J)}{(\mathcal{D} \otimes (\bigoplus_{J \subseteq I} \bar{\Omega}_J \otimes V_J)) \cap \mathcal{J}}.$$

We have $\text{gr}^F \mathcal{D}_{\bar{z}_I} \cong \mathcal{O}_{\bar{z}_I}[\xi_1, \dots, \xi_n]$ and M is a finitely generated $\mathcal{O}_{\bar{z}_I}[\xi_1, \dots, \xi_n]$ -module. We denote by $\mathcal{M}ax$ the maximal ideal of the local ring $\mathcal{O}_{\bar{z}_I}$. Let

$$Q_{\tilde{\xi}_I} := \mathcal{M}ax + (\xi_1 - \tilde{\xi}_1, \dots, \xi_n - \tilde{\xi}_n).$$

This defines a maximal ideal in $\mathcal{O}_{\bar{z}_I}[\xi_1, \dots, \xi_n]$. Thus, $M/Q_{\tilde{\xi}_I} M$ is a finitely generated $\mathcal{O}_{\bar{z}_I}[\xi_1, \dots, \xi_n]/Q_{\tilde{\xi}_I}$ -vector space. Therefore, there exists a polynomial $P_{M, Q_{\tilde{\xi}_I}}(N)$, called Hilbert-Samuel polynomial, and an integer $N_0 \in \mathbb{N}$ such that

$$P_{M, Q_{\tilde{\xi}_I}}(N) = \text{length}(M/Q_{\tilde{\xi}_I}^N M) \quad \text{for all } N \geq N_0.$$

The highest degree term of P has the form $\frac{e}{d!} N^d$ where $e \in \mathbb{N}$, $d \in \mathbb{N}$ and by definition

$$e = \text{mult}_{\bar{z}_I}^* EV.$$

Applying [Ser75, Proposition 11a) in Subsection II.B)4], we receive

$$P_{M, Q_{\tilde{\xi}_I}}(N) = P_{T^{-1}M, T^{-1}Q_{\tilde{\xi}_I}}(N) \quad \text{for } T := \mathcal{O}_{\bar{z}_I}[\xi_1, \dots, \xi_n] \setminus Q_{\tilde{\xi}_I}.$$

So we need to consider the localisation of M at T :

$$T^{-1}M = \bigoplus_{k \in \mathbb{N}_0} T^{-1}(\text{gr}_k^F EV)_{\bar{z}_I}$$

Let $[P \otimes \omega_K \otimes v_K]$ denote the image of $P \otimes \omega_K \otimes v_K \in F_k \mathcal{D}_{\bar{z}_I} \otimes \bar{\Omega}_K \otimes V_K$ in $(\text{gr}_k^F EV)_{\bar{z}_I}$ for $K \subsetneq I$. For every $i \in I \setminus K$ we have the following identity in $(\text{gr}_{k+1}^F EV)_{\bar{z}_I}$:

$$\xi_i \cdot [P \otimes \omega_K \otimes v_K] = [P \otimes \omega_{K \cup \{i\}} \otimes B_{K \cup \{i\}, K}(v_K)] = 0$$

Consider this identity in $T^{-1}M$: The map $\xi_i \cdot _ : T^{-1}(\text{gr}_k^F EV)_{\bar{z}_I} \rightarrow T^{-1}(\text{gr}_{k+1}^F EV)_{\bar{z}_I}$ is

bijjective for $i \in I$, as $\tilde{\xi}_i \neq 0$ for $i \in I$. Therefore, $\frac{[P \otimes \omega_K \otimes v_K]}{1} = 0$ and we may assume that

$$T^{-1}(F_k E\mathcal{V})_{\tilde{z}_I} = \frac{T^{-1}F_k \mathcal{D}_{\tilde{z}_I} \otimes \overline{\Omega}_I \otimes V_I}{\left(T^{-1}F_k \mathcal{D}_{\tilde{z}_I} \otimes \overline{\Omega}_I \otimes V_I\right) \cap T^{-1}\mathcal{J}_{\tilde{z}_I}}.$$

Using [Ser75, Proposition 11a)] the other way round, we may assume that

$$(F_k E\mathcal{V})_{\tilde{z}_I} = \frac{F_k \mathcal{D}_{\tilde{z}_I} \otimes \overline{\Omega}_I \otimes V_I}{\left(F_k \mathcal{D}_{\tilde{z}_I} \otimes \overline{\Omega}_I \otimes V_I\right) \cap \mathcal{J}_{\tilde{z}_I}} \quad \text{or} \quad E\mathcal{V} = \frac{\mathcal{D} \otimes \overline{\Omega}_I \otimes V_I}{\left(\mathcal{D} \otimes \overline{\Omega}_I \otimes V_I\right) \cap \mathcal{J}}.$$

For simplicity let $I = \{1, \dots, |I|\}$ for the moment. Set $n_I := \dim_{\mathbb{C}}(V_I)$. Consider the following exact sequence of holonomic \mathcal{D} -modules

$$\begin{aligned} 0 \longrightarrow \ker(\pi) \hookrightarrow \widetilde{\mathcal{N}} \xrightarrow{\pi} E\mathcal{V} \longrightarrow 0 \\ \text{where } \widetilde{\mathcal{N}} := \mathcal{D} \otimes V_I / \left(z_1 \otimes V_I, \dots, z_{|I|} \otimes V_I, \partial_{|I|+1} \otimes V_I, \dots, \partial_n \otimes V_I \right) \cong \\ \cong \bigoplus_{n_I\text{-times}} \mathcal{D} / (z_1, \dots, z_{|I|}, \partial_{|I|+1}, \dots, \partial_n) =: \bigoplus_{n_I\text{-times}} \mathcal{N}. \end{aligned}$$

This sequence yields $\text{mult}_{\tilde{z}_I^*} E\mathcal{V} \leq \text{mult}_{\tilde{z}_I^*} \widetilde{\mathcal{N}}$. Furthermore, $\text{mult}_{\tilde{z}_I^*} \widetilde{\mathcal{N}} = n_I \cdot \text{mult}_{\tilde{z}_I^*} \mathcal{N}$. So let us compute $\text{mult}_{\tilde{z}_I^*} \mathcal{N}$ where $\tilde{z}_I^* = (0, \dots, 0, \tilde{z}_{|I|+1}, \dots, \tilde{z}_n, \tilde{\xi}_1, \dots, \tilde{\xi}_{|I|}, 0, \dots, 0) =: (\tilde{z}_I, \tilde{\xi}_I)$ with $\tilde{z}_{|I|+1}, \dots, \tilde{z}_n, \tilde{\xi}_1, \dots, \tilde{\xi}_{|I|} \neq 0$:

We use the good filtration $F_{\bullet} \mathcal{N}$ on \mathcal{N} which is induced by the standard filtration $F_{\bullet} \mathcal{D}$ of \mathcal{D} . So, we consider $(\text{gr}^F \mathcal{N})_{\tilde{z}_I} \cong \mathbb{C}\{z_{|I|+1} - \tilde{z}_{|I|+1}, \dots, z_n - \tilde{z}_n\}[\xi_1, \dots, \xi_{|I|}]$ as a module over $(\text{gr}^F \mathcal{D})_{\tilde{z}_I} \cong \mathbb{C}\{z_1, \dots, z_{|I|}, z_{|I|+1} - \tilde{z}_{|I|+1}, \dots, z_n - \tilde{z}_n\}[\xi_1, \dots, \xi_n]$. Let $\mathcal{M}ax$ be the maximal ideal of $\mathbb{C}\{z_1, \dots, z_{|I|}, z_{|I|+1} - \tilde{z}_{|I|+1}, \dots, z_n - \tilde{z}_n\}$. We need to compute the multiplicity of $(\text{gr}^F \mathcal{N})_{\tilde{z}_I}$ with respect to the maximal ideal $\mathcal{M}ax + (\xi_1 - \tilde{\xi}_1, \dots, \xi_{|I|} - \tilde{\xi}_{|I|}, \xi_{|I|+1}, \dots, \xi_n)$ of $(\text{gr}^F \mathcal{D})_{\tilde{z}_I}$. A shift of coordinates gives us that we equivalently have to treat

$$\mathbb{C}\{z_{|I|+1}, \dots, z_n\}[\xi_1, \dots, \xi_{|I|}] \quad \text{as a module over} \quad \mathbb{C}\{z_1, \dots, z_n\}[\xi_1, \dots, \xi_n],$$

and compute its multiplicity with respect to the maximal ideal

$$Q := (z_1, \dots, z_n, \xi_1 - \tilde{\xi}_1, \dots, \xi_{|I|} - \tilde{\xi}_{|I|}, \xi_{|I|+1}, \dots, \xi_n).$$

So we have to compute

$$\text{length} \left(\frac{\mathbb{C}\{z_{|I|+1}, \dots, z_n\}[\xi_1, \dots, \xi_{|I|}]}{(z_{|I|+1}, \dots, z_n, \xi_1 - \tilde{\xi}_1, \dots, \xi_{|I|} - \tilde{\xi}_{|I|})^N \cdot \mathbb{C}\{z_{|I|+1}, \dots, z_n\}[\xi_1, \dots, \xi_{|I|}]} \right).$$

But this is the number of monomials of degree less than N in $\mathbb{C}\{z_{|I|+1}, \dots, z_n\}[\xi_1, \dots, \xi_{|I|}]$ which is equal to $\binom{N-1+n}{N-1}$. This shows us that $\text{mult}_{\tilde{z}_I^*} \mathcal{N} = 1$ and $\text{mult}_{\tilde{z}_I^*} E\mathcal{V} \leq n_I$.

- (iv) Now, we construct the canonical isomorphism η_I from V_I^* into (1). For this purpose let $\alpha \in V_I^*$. We define $\eta_I(\alpha)$ as follows:

Let $\{m_1, \dots, m_{|I|}\} = I$, $\{l_1, \dots, l_{n-|I|}\} = \{1, \dots, n\} \setminus I$. For a moment fix a basis of V_I and denote it by $v_{I,1}, \dots, v_{I,n_I}$. In abuse of notation we denote the matrices corresponding to the maps α , $\mathcal{B}_{I,I \cup \{l\}} := B_{I,I \cup \{l\}} B_{I \cup \{l\},I}$ and $\mathcal{B}_{I,I \setminus \{m\}} := B_{I,I \setminus \{m\}} B_{I \setminus \{m\},I}$ w.r. t. this basis by the same symbols. We set

$$\mathcal{F} := z_{l_1}^{\mathcal{B}_{I,I \cup \{l_1\}}} \cdot \dots \cdot z_{l_{n-|I|}}^{\mathcal{B}_{I,I \cup \{l_{n-|I|\}}}} \cdot z_{m_1}^{\mathcal{B}_{I,I \setminus \{m_1\}} - \text{Id}} \cdot \dots \cdot z_{m_{|I|}}^{\mathcal{B}_{I,I \setminus \{m_{|I|\}} - \text{Id}}} \quad \text{and} \\ \eta_I(\alpha) := \alpha \cdot \mathcal{F}.$$

One verifies directly that $\eta_I(\alpha)$ is indeed an element in (1) by plugging it into (\star) .

We need to verify that this construction of η_I is independent of the choice of basis of V_I . So, let $\tilde{v}_{I,1}, \dots, \tilde{v}_{I,n_I}$ denote another basis of V_I . Let $\tilde{\mathcal{B}}_{I,I \cup \{l\}}$, $\tilde{\mathcal{B}}_{I,I \setminus \{m\}}$ and $\tilde{\alpha}$ denote the matrices corresponding to the linear maps $\mathcal{B}_{I,I \cup \{l\}}$, $\mathcal{B}_{I,I \setminus \{m\}}$ and α w.r. t. this new basis. Let R denote the matrix of the change of coordinates from $\{v_{I,1}, \dots, v_{I,n_I}\}$ to $\{\tilde{v}_{I,1}, \dots, \tilde{v}_{I,n_I}\}$. Let $v_I \in V_I$. We denote by v_I in abuse of notation the vector w.r. t. the basis $\{v_{I,1}, \dots, v_{I,n_I}\}$ and by \tilde{v}_I the vector w.r. t. the basis $\{\tilde{v}_{I,1}, \dots, \tilde{v}_{I,n_I}\}$. We receive

$$\begin{aligned} \eta_I(\alpha)(v_I) &= \alpha \cdot z_{l_1}^{\mathcal{B}_{I,I \cup \{l_1\}}} \cdot \dots \cdot z_{l_{n-|I|}}^{\mathcal{B}_{I,I \cup \{l_{n-|I|\}}}} \cdot z_{m_1}^{\mathcal{B}_{I,I \setminus \{m_1\}} - \text{Id}} \cdot \dots \cdot z_{m_{|I|}}^{\mathcal{B}_{I,I \setminus \{m_{|I|\}} - \text{Id}}} \cdot v_I = \\ &= \tilde{\alpha} R R^{-1} z_{l_1}^{\tilde{\mathcal{B}}_{I,I \cup \{l_1\}}} R \dots R^{-1} z_{l_{n-|I|}}^{\tilde{\mathcal{B}}_{I,I \cup \{l_{n-|I|\}}}} \tilde{z}_{m_1}^{\tilde{\mathcal{B}}_{I,I \setminus \{m_1\}} - \text{Id}} R \dots R^{-1} \tilde{z}_{m_{|I|}}^{\tilde{\mathcal{B}}_{I,I \setminus \{m_{|I|\}} - \text{Id}}} R R^{-1} \tilde{v}_I = \\ &= \eta_I(\tilde{\alpha})(\tilde{v}_I). \end{aligned}$$

Hence, our construction is independent of the choice of basis of V_I .

Now, we want to check that η_I is injective. So assume that $\eta_I(\alpha)$ is the zero mapping. As ∂_m acts bijective on $\mathcal{O}_{I,0}$ for $m \in I$ (see Lemma 2.8), this is equivalent to

$$\partial_{m_1}^{-1} \dots \partial_{m_{|I|}}^{-1} z_{l_1}^{\mathcal{B}_{I,I \cup \{l_1\}}^T} \cdot \dots \cdot z_{l_{n-|I|}}^{\mathcal{B}_{I,I \cup \{l_{n-|I|\}}^T}} \cdot z_{m_1}^{\mathcal{B}_{I,I \setminus \{m_1\}}^T - \text{Id}} \cdot \dots \cdot z_{m_{|I|}}^{\mathcal{B}_{I,I \setminus \{m_{|I|\}}^T - \text{Id}}} \cdot \alpha^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The eigenvalues of $\mathcal{B}_{I,I \setminus \{m\}}^T - \text{Id}$ are contained in Σ for $m \in I$, as \mathcal{V} is an object in $\mathcal{Q}ui_n^{\Sigma_1}$. Using Lemma 2.9, we receive $\alpha \equiv 0$ and η_I is injective.

As $\dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}_0}((E\mathcal{V})_0, \mathcal{O}_{I,0}) \leq n_I$ by part (iv), we immediately receive the bijectivity of η_I as claimed.

Composing the isomorphism from part (i) with (ALG), we receive a natural isomorphism from (1) into $\text{Hom}_{\mathcal{D}_0}((E\mathcal{V})_0, \mathcal{O}_{I,0})$. Composing this with the isomorphism η_I from V_I^* into (1), this gives us the canonical isomorphism

$$\mathfrak{a}: V_I^* \xrightarrow{\cong} \text{Hom}_{\mathcal{D}_0}((E\mathcal{V})_0, \mathcal{O}_{I,0}). \quad \square$$

The following statement on the matrix polynomial will be used in the proof of Proposition 2.12.

Corollary 2.11. *Let A denote a square matrix with entries in \mathbb{C} and let $i \in \{1, \dots, n\}$. We fix a branch of the logarithm defined on $\mathbb{C}_i \setminus \mathbb{R}_0^+$ and let $z_i^A = \exp(A \cdot \ln(z_i))$ as before. Set*

$$\varphi_A(z_i) := \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} \cdot \ln(z_i)^{k+1} \quad \text{and} \quad \psi(A) := \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} A^{k-1}.$$

Then

$$M_i \varphi_A(z_i) - \varphi_A(z_i) = \psi(A) \cdot z_i^A.$$

Proof. We have

$$M_i \varphi_A(z_i) = \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} \cdot (\ln(z_i) + 2\pi i)^{k+1}.$$

Furthermore direct computation yields

$$A \cdot (M_i \varphi_A(z_i) - \varphi_A(z_i)) = (M_i \varphi_A(z_i) - \varphi_A(z_i)) \cdot A = \psi(A) \cdot z_i^A \cdot A = A \cdot \psi(A) \cdot z_i^A.$$

We may assume for our arguments that $A = J_a$ where J_a is a single Jordan block with eigenvalue $a \in \mathbb{C}$. If $a \neq 0$, our claim follows directly. So assume $a = 0$. The above equation shows us however that $M_i \varphi_A(z_i) - \varphi_A(z_i)$ and $\psi(A) \cdot z_i^A$ coincide up to a possible difference in the entry in the upper-left corner. The entry of $M_i \varphi_A(z_i) - \varphi_A(z_i)$ in the upper-left corner is $\ln(z_i) + 2\pi i - \ln(z_i) = 2\pi i$. The first column of z_i^A is $(1, 0, \dots, 0)^T$ and the entry in the upper-left corner of $\psi(A)$ is $2\pi i$. Hence, the entry in the upper-left corner of $\psi(A) \cdot z_i^A$ is $2\pi i$ as well which proves the claim. \square

In the following we prove that the quiver representation one receives after applying \mathcal{A} to a quiver \mathcal{D} -module is determined in a simple manner by the starting quiver representation. To do so, we “extend” the canonical isomorphism \mathfrak{a} from Proposition 2.10 to the whole quiver representation.

Proposition 2.12. *Let $\mathcal{V}_n = (V_I, B_{I \cup \{i\}, I}, B_{I, I \cup \{i\}})$ be an object in $\text{Qui}_n^{\Sigma_1}$ and EV_n the corresponding quiver \mathcal{D} -module. The image of EV_n under the functor \mathcal{A} is canonically isomorphic to*

$$V_I^* \xleftarrow[w_{I,i}]{u_{I,i}} V_{I \cup \{i\}}^*$$

where

$$u_{I,i} = B_{I, I \cup \{i\}}^* \quad \text{and} \quad w_{I,i} = B_{I \cup \{i\}, I}^* \circ \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} (B_{I, I \cup \{i\}}^* \circ B_{I \cup \{i\}, I}^*)^{k-1}.$$

Proof. Let $n_I = \dim_{\mathbb{C}} V_I$ as before. Set $\mathcal{B}_{K,L} := B_{K,L} \circ B_{L,K}$ if $K, L \in \mathcal{P}(\{1, \dots, n\})$ are adjacent, i. e. $K = L \cup \{l\}$ or $L = K \cup \{k\}$. The image of $(V_I, B_{I \cup \{i\}, I}, B_{I, I \cup \{i\}})$ under $\mathcal{A} \circ E$ is given by

$$(\text{Hom}_{\mathcal{D}_0}((EV_n)_0, \mathcal{O}_{I,0}), \text{can}_{I,i}, \text{var}_{I,i}).$$

First, we reperform the first steps of the proof of Proposition 2.10. Then we compute can and var .

- (i) First, note that the natural isomorphism we gave for $\text{Hom}_{\mathcal{D}_0}((EV_n)_0, \mathcal{O}_{I,0})$ in part (i) of the proof of Proposition 2.10 is compatible with the canonical map and the variation. Therefore,

it extends to the entire object $(\text{Hom}_{\mathcal{D}_0}((E\mathcal{V}_n)_0, \mathcal{O}_{I,0}), \text{can}_{I,i}, \text{var}_{I,i})$. We receive:

$$\begin{aligned}
& \text{Hom}_{\mathcal{D}_0}((E\mathcal{V}_n)_0, \mathcal{O}_{I,0}) \xrightleftharpoons[\text{var}_{I,i}]{\text{can}_{I,i}} \text{Hom}_{\mathcal{D}_0}((E\mathcal{V}_n)_0, \mathcal{O}_{I \cup \{i\},0}) \cong \\
& \left\{ \bigoplus_J \phi_I^J \in \bigoplus_J \text{Hom}_{\mathbb{C}}(V_J, \mathcal{O}_{I,0}) \mid \partial_j \cdot \phi_I^J(v_J) - \phi_I^{J \cup \{j\}}(B_{J \cup \{j\},J}(v_J)) = 0, \right. \\
& \quad \left. z_j \cdot \phi_I^{J \cup \{j\}}(v_{J \cup \{j\}}) - \phi_I^J(B_{J, J \cup \{j\}}(v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \dots, n\}, j \in \{1, \dots, n\} \setminus J \right\} \\
& \quad \bigoplus_J \text{can}_{I,i}^J \downarrow \quad \uparrow \quad \bigoplus_J \text{var}_{I,i}^J \\
& \left\{ \bigoplus_J \phi_{I \cup \{i\}}^J \in \bigoplus_J \text{Hom}_{\mathbb{C}}(V_J, \mathcal{O}_{I \cup \{i\},0}) \mid \partial_j \cdot \phi_{I \cup \{i\}}^J(v_J) - \phi_{I \cup \{i\}}^{J \cup \{j\}}(B_{J \cup \{j\},J}(v_J)) = 0, \right. \\
& \quad \left. z_j \cdot \phi_{I \cup \{i\}}^{J \cup \{j\}}(v_{J \cup \{j\}}) - \phi_{I \cup \{i\}}^J(B_{J, J \cup \{j\}}(v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \dots, n\}, j \in \{1, \dots, n\} \setminus J \right\}
\end{aligned}$$

- (ii) Let us fix $I \in \mathcal{P}(\{1, \dots, n\}) \setminus \{1, \dots, n\}$, $i \in \{1, \dots, n\} \setminus I$ temporarily. We may consider only the behaviour under the canonical map and the variation of the two pairs

$$\begin{pmatrix} \phi_I^I \\ \phi_{I \cup \{i\}}^{I \cup \{i\}} \end{pmatrix} \leftrightarrow \begin{pmatrix} \phi_{I \cup \{i\}}^I \\ \phi_{I \cup \{i\}}^{I \cup \{i\}} \end{pmatrix}.$$

For any $K \in \mathcal{P}(\{1, \dots, n\})$, $\phi_I^K \leftrightarrow \phi_{I \cup \{i\}}^K$ will follow their behaviour under these two maps. This can be seen by adapting (ALG) in the following way: We only use equations of (\star) which involve z_j for $j \notin I \cup \{i\}$, or ∂_k for $k \in I$. That way we express ϕ_I^K in terms of ϕ_I^I if $i \notin K$, or in terms of $\phi_I^{I \cup \{i\}}$ if $i \in K$. This expression is unique with the same arguments as for (ALG). One observes that $\phi_I^K, \phi_{I \cup \{i\}}^K$ are build up from $\phi_I^I, \phi_{I \cup \{i\}}^I$ if $i \notin K$ (from $\phi_I^{I \cup \{i\}}$, $\phi_{I \cup \{i\}}^{I \cup \{i\}}$ if $i \in K$) in a completely identical manner. This ensures that they behave in the same way under the canonical map and the variation.

- (iii) Using the isomorphisms η_I and $\eta_{I \cup \{i\}}$ from part (iv) of the proof of Proposition 2.10, we can uniquely identify ϕ_I^I and $\phi_{I \cup \{i\}}^{I \cup \{i\}}$ with elements of V_I^* and $V_{I \cup \{i\}}^*$, respectively. After a choice of basis of V_I and $V_{I \cup \{i\}}$, we may write for some $\alpha_I \in V_I^*$ and $\alpha_{I \cup \{i\}} \in V_{I \cup \{i\}}^*$ (we omit set braces for singletons in the following)

$$\begin{aligned}
\phi_I^I &= \eta_I(\alpha_I) = \alpha_I \cdot \mathcal{F}_I & \phi_{I \cup \{i\}}^I &= \partial_i^{-1} \cdot \phi_{I \cup \{i\}}^{I \cup \{i\}} \cdot B_{I \cup \{i\}, I} \\
\phi_I^{I \cup \{i\}} &= z_i^{-1} \cdot \phi_I^I \cdot B_{I, I \cup \{i\}} & \phi_{I \cup \{i\}}^{I \cup \{i\}} &= \eta_{I \cup \{i\}}(\alpha_{I \cup \{i\}}) = \alpha_{I \cup \{i\}} \cdot \mathcal{F}_{I \cup \{i\}} \\
\mathcal{F}_I &= z_i^{\mathcal{B}_{I, I \cup \{i\}}} \cdot z_{l_2}^{\mathcal{B}_{I, I \cup \{l_2\}}} \cdot \dots \cdot z_{l_{n-|I|}}^{\mathcal{B}_{I, I \cup \{l_{n-|I|\}}} \cdot z_{m_1}^{\mathcal{B}_{I, I \setminus m_1} - \text{Id}} \cdot \dots \cdot z_{m_{|I|}}^{\mathcal{B}_{I, I \setminus m_{|I|}} - \text{Id}} \\
\mathcal{F}_{I \cup \{i\}} &= z_{l_2}^{\mathcal{B}_{I \cup \{i\}, I \cup \{l_2\}}} \cdot \dots \cdot z_{l_{n-|I|}}^{\mathcal{B}_{I \cup \{i\}, I \cup \{l_{n-|I|\}}} \cdot z_i^{\mathcal{B}_{I \cup \{i\}, I} - \text{Id}} \cdot z_{m_1}^{\mathcal{B}_{I \cup \{i\}, \{I \cup \{i\}\} \setminus m_1} - \text{Id}} \cdot \dots \cdot z_{m_{|I|}}^{\mathcal{B}_{I \cup \{i\}, \{I \cup \{i\}\} \setminus m_{|I|}} - \text{Id}}
\end{aligned}$$

where $\{i, l_2, \dots, l_{n-|I|}\} = \{1, \dots, n\} \setminus I$, $\{m_1, \dots, m_{|I|}\} = I$. This description of ϕ^\bullet is independent of the choice of basis as we showed in part (iv) of the proof of Proposition 2.10.

Let us give some helpful identities for the computations. We have for $i, l \notin I$, $l \neq i$, $m \in I$:

$$\begin{aligned} B_{I, I \cup i} \cdot \mathcal{B}_{I \cup i, \{I \cup i\} \setminus m} &= \mathcal{B}_{I, I \setminus m} \cdot B_{I, I \cup i} & B_{I \cup i, I} \cdot \mathcal{B}_{I, I \cup l} &= \mathcal{B}_{I \cup i, I \cup \{i, l\}} \cdot B_{I \cup i, I} \\ B_{I, I \cup i} \cdot \mathcal{B}_{I \cup i, I \cup \{i, l\}} &= \mathcal{B}_{I, I \cup l} \cdot B_{I, I \cup i} & B_{I \cup i, I} \cdot \mathcal{B}_{I, I \setminus m} &= \mathcal{B}_{I \cup i, \{I \cup i\} \setminus m} \cdot B_{I \cup i, I} \end{aligned}$$

- (iv) We claimed that the canonical map from $\begin{pmatrix} \phi_I^I \\ \phi_{I \cup i}^{I \cup i} \end{pmatrix}$ to $\begin{pmatrix} \phi_{I \cup i}^I \\ \phi_{I \cup i}^{I \cup i} \end{pmatrix}$ is given by $B_{I, I \cup i}^*$. This means we have to check that the assignment

$$\alpha_I \mapsto \alpha_{I \cup i} := \alpha_I \cdot B_{I, I \cup i}$$

describes the canonical map. This follows by direct computations:

$$\eta_{I \cup i}(\alpha_I \cdot B_{I, I \cup i}) = \alpha_I \cdot B_{I, I \cup i} \cdot \mathcal{F}_{I \cup i} = z_i^{-1} \cdot \alpha_I \cdot \mathcal{F}_I \cdot B_{I, I \cup i} = z_i^{-1} \cdot \phi_I^I \cdot B_{I, I \cup i} = \phi_{I \cup i}^{I \cup i}$$

and therefore

$$\partial_i^{-1} \cdot \eta_{I \cup i}(\alpha_I \cdot B_{I, I \cup i}) \cdot B_{I \cup i, I} = \partial_i^{-1} z_i^{-1} \cdot \alpha_I \cdot \mathcal{F}_I \cdot B_{I, I \cup i} \cdot B_{I \cup i, I} = \alpha_I \cdot \mathcal{F}_I = \phi_I^I$$

With the same arguments as before, one can show that the description of the canonical map is independent of the choice of basis.

- (v) We are left with the computation of the variation $M_i \phi_{I \cup i}^I - \phi_{I \cup i}^I$ and $M_i \phi_{I \cup i}^{I \cup i} - \phi_{I \cup i}^{I \cup i}$. In particular, we need to check that the assignment

$$\alpha_{I \cup i} \mapsto \alpha_I := \alpha_{I \cup i} \cdot \left(\sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} (B_{I \cup i, I} B_{I, I \cup \{i\}})^{k-1} \right) \cdot B_{I \cup i, I} = \alpha_{I \cup i} \cdot \underbrace{\psi(\mathcal{B}_{I \cup i, I}) \cdot B_{I \cup i, I}}_{=: \Theta_{I, i}}$$

describes the variation. For $\phi_{I \cup i}^{I \cup i}$ the correctness follows by direct computation:

$$\begin{aligned} M_i \phi_{I \cup i}^{I \cup i} - \phi_{I \cup i}^{I \cup i} &= \alpha_{I \cup i} \cdot (e^{2\pi i \mathcal{B}_{I \cup i, I}} - \text{Id}) \cdot \mathcal{F}_{I \cup i} = \alpha_{I \cup i} \cdot \Theta_{I, i} \cdot B_{I, I \cup i} \cdot \mathcal{F}_{I \cup i} = \\ &= z_i^{-1} \cdot \alpha_{I \cup i} \cdot \Theta_{I, i} \cdot \mathcal{F}_I \cdot B_{I, I \cup i} = z_i^{-1} \cdot \eta_I(\alpha_{I \cup i} \cdot \Theta_{I, i}) \cdot B_{I, I \cup i} \end{aligned}$$

Now, let us compute $M_i \phi_{I \cup i}^I - \phi_{I \cup i}^I$: We use the identity $\mathcal{F}_{I \cup i} \cdot B_{I \cup i, I} = z_i^{-1} \cdot B_{I \cup i, I} \cdot \mathcal{F}_I$ to rearrange $\phi_{I \cup i}^I$. We receive

$$\begin{aligned} \phi_{I \cup i}^I &= \alpha_{I \cup i} B_{I \cup i, I} \cdot \left(\partial_i^{-1} z_i^{-1} z_i^{\mathcal{B}_{I, I \cup i}} \right) \cdot z_{l_2}^{\mathcal{B}_{I, I \cup l_2}} \cdot \dots \cdot z_{l_{n-|I|}}^{\mathcal{B}_{I, I \cup l_{n-|I|}}} \cdot z_{m_1}^{\mathcal{B}_{I, I \setminus m_1} - \text{Id}} \cdot \dots \cdot z_{m_{|I|}}^{\mathcal{B}_{I, I \setminus m_{|I|}} - \text{Id}} \\ \text{where } \partial_i^{-1} z_i^{-1} z_i^{\mathcal{B}_{I, I \cup i}} &= \sum_{k=0}^{\infty} \frac{\mathcal{B}_{I, I \cup i}^k}{k!} \frac{\ln(z_i)^{k+1}}{k+1} =: \varphi_{\mathcal{B}_{I, I \cup i}}(z_i). \end{aligned}$$

Corollary 2.11 yields

$$M_i \varphi_{\mathcal{B}_{I, I \cup i}}(z_i) - \varphi_{\mathcal{B}_{I, I \cup i}}(z_i) = \left(\sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} \cdot \mathcal{B}_{I, I \cup i}^{k-1} \right) \cdot z_i^{\mathcal{B}_{I, I \cup i}}.$$

This gives us

$$\begin{aligned}
M_i \phi_{I \cup i}^I - \phi_{I \cup i}^I &= \\
&= \alpha_{I \cup i} B_{I \cup i, I} \left(\sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} \cdot \mathcal{B}_{I, I \cup i}^{k-1} \right) z_i^{\mathcal{B}_{I, I \cup i}} z_{l_2}^{\mathcal{B}_{I, I \cup l_2}} \dots z_{l_{n-|I|}}^{\mathcal{B}_{I, I \cup l_{n-|I|}}} z_{m_1}^{\mathcal{B}_{I, I \setminus m_1} - \text{Id}} \dots z_{m_{|I|}}^{\mathcal{B}_{I, I \setminus m_{|I|}} - \text{Id}} = \\
&= \alpha_{I \cup i} \cdot \Theta_{I, i} \cdot \mathcal{F}_I = \eta_I(\alpha_{I \cup i} \cdot \Theta_{I, i}).
\end{aligned}$$

Once more, note that these computations are independent of the choice of basis. \square

Now, we have collected all the important pieces for the proof of our Main Theorem 2.7:

Proof of Theorem 2.7. We need to examine if the family of isomorphisms from Proposition 2.12 is natural: The isomorphism we gave in part (i) of the proof of Proposition 2.12 is natural. So let $\mathcal{V} = (V_J, B_{J \cup \{j\}, J}, B_{J, J \cup \{j\}})$ and $\tilde{\mathcal{V}} = (\tilde{V}_J, \tilde{B}_{J \cup \{j\}, J}, \tilde{B}_{J, J \cup \{j\}})$ denote two objects in $\mathcal{Q}ui_n^{\Sigma^1}$ and let $\tau = (h_J)$ denote a morphism from \mathcal{V} to $\tilde{\mathcal{V}}$. We need to check that the diagram

$$\begin{array}{ccc}
\{\oplus_J \phi_I^J \in \oplus_J \text{Hom}_{\mathbb{C}}(V_J, \mathcal{O}_{I,0}) \mid \dots\} & \xleftarrow{(\text{Hom}_{\mathbb{C}}(\tau, \mathcal{O}_{I,0}))} & \{\oplus_J \tilde{\phi}_I^J \in \oplus_J \text{Hom}_{\mathbb{C}}(\tilde{V}_J, \mathcal{O}_{I,0}) \mid \dots\} \\
\oplus_J \text{can}_{I,i}^J \downarrow \uparrow \oplus_J \text{var}_{I,i}^J & & \oplus_J \tilde{\text{can}}_{I,i}^J \downarrow \uparrow \oplus_J \tilde{\text{var}}_{I,i}^J \\
\{\oplus_J \phi_{I \cup \{i\}}^J \in \oplus_J \text{Hom}_{\mathbb{C}}(V_J, \mathcal{O}_{I \cup \{i\},0}) \mid \dots\} & & \{\oplus_J \tilde{\phi}_{I \cup \{i\}}^J \in \oplus_J \text{Hom}_{\mathbb{C}}(\tilde{V}_J, \mathcal{O}_{I \cup \{i\},0}) \mid \dots\} \\
\uparrow & & \uparrow \\
\text{Hom}_{\mathbb{C}}(V_I, \mathbb{C}) & & \text{Hom}_{\mathbb{C}}(\tilde{V}_I, \mathbb{C}) \\
B_{I, I \cup \{i\}}^* \downarrow \uparrow \psi(B_{I, I \cup \{i\}}^*) \circ B_{I \cup \{i\}, I}^* & \xleftarrow{(h_I^*)} & \tilde{B}_{I, I \cup \{i\}}^* \downarrow \uparrow \psi(\tilde{B}_{I, I \cup \{i\}}^*) \circ \tilde{B}_{I \cup \{i\}, I}^* \\
\text{Hom}_{\mathbb{C}}(V_{I \cup \{i\}}, \mathbb{C}) & & \text{Hom}_{\mathbb{C}}(\tilde{V}_{I \cup \{i\}}, \mathbb{C})
\end{array}$$

commutes. The properties indicated by “...” may be found in part (i) of the proof of Proposition 2.12. The morphisms in the horizontal rows are given by

$$\begin{aligned}
h_I^*: \text{Hom}_{\mathbb{C}}(\tilde{V}_I, \mathbb{C}) &\rightarrow \text{Hom}_{\mathbb{C}}(V_I, \mathbb{C}), & \tilde{\alpha}_I &\rightarrow \tilde{\alpha}_I \circ h_I \\
\text{Hom}_{\mathbb{C}}((h_J), \mathcal{O}_{I,0}): \bigoplus_J \text{Hom}_{\mathbb{C}}(\tilde{V}_J, \mathcal{O}_{I,0}) &\rightarrow \bigoplus_J \text{Hom}_{\mathbb{C}}(V_J, \mathcal{O}_{I,0}), & \bigoplus_J \tilde{\alpha}_I^J &\rightarrow \bigoplus_J (\tilde{\alpha}_I^J \circ h_J).
\end{aligned}$$

The isomorphisms from the lower row into the upper row are given by (ALG) composed with (η_I) and $(\tilde{\eta}_I)$, respectively. The commutativity of the diagram follows now easily using the commutativity conditions of the morphism (h_I) with the $B_{\bullet, \bullet}$ and $\tilde{B}_{\bullet, \bullet}$ -maps. Hence, the diagram of Theorem 2.7 commutes up to a natural isomorphism. The remaining claims follow directly from that. \square

References

- [Bjö93] J.-E. Björk. *Analytic \mathcal{D} -modules and applications*, volume 247 of *Mathematics and its applications*. Kluwer Academic Publishers, Dordrecht, 1993.
- [Dim04] A. Dimca. *Sheaves in topology*. Universitext. Springer-Verlag, Berlin, 2004.
- [GGM85a] A. Galligo, M. Granger, and Ph. Maisonobe. \mathcal{D} -modules et faisceaux pervers dont le support singulier est un croisement normal. *Ann. Inst. Fourier (Grenoble)*, 35(1):1–48, 1985.
- [GGM85b] A. Galligo, M. Granger, and Ph. Maisonobe. \mathcal{D} -modules et faisceaux pervers dont le support singulier est un croisement normal. II. *Astérisque*, (130):240–259, 1985.
- [GM93] M. Granger and Ph. Maisonobe. A basic course on differential modules. In *Éléments de la théorie des systèmes différentiels. \mathcal{D} -modules cohérents et holonomes*, volume 45 of *Travaux en Cours*, pages 103–168. Hermann, Paris, 1993.
- [HJ91] R. A. Horn and C. R. Johnson. *Topics in matrix analysis*. Cambridge University Press, Cambridge, 1991.
- [Kas84] M. Kashiwara. The Riemann-Hilbert problem for holonomic systems. *Publ. Res. Inst. Math. Sci.*, 20(2):319–365, 1984.
- [Kas03] M. Kashiwara. *D -modules and microlocal calculus*, volume 217 of *Translations of mathematical monographs*. American Mathematical Society, Providence, R.I., 2003.
- [KV06] S. Khoroshkin and A. Varchenko. Quiver D -modules and homology of local systems over an arrangement of hyperplanes. *IMRP Int. Math. Res. Pap.*, 2006:1–116, 2006. Art. ID 69590.
- [Mal91] B. Malgrange. *Equations différentielles à coefficients polynomiaux*, volume 96 of *Progress in mathematics*. Birkhäuser, Boston, Mass., 1991.
- [Meb84] Z. Mebkhout. Une équivalence de catégories, Une autre équivalence de catégories. *Compos. Math.*, 51(1):51–62, 63–88, 1984.
- [Ser75] J.-P. Serre. *Algèbre locale, multiplicités*, volume 11 of *Lecture notes in mathematics*. Springer-Verlag, Berlin, 3rd edition, 1975.

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